This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

HARVARD COLLEGE
LIBRARY

THE ESSEX INSTITUTE
TEXT-BOOK COLLECTION

GIFT OF
GEORGE ARTHUR PLIMPTON
OF NEW YORK

JANUARY 25, 1924

3 2044 097 046 288

Eaton's Mathematical Series.

# KEY

TO

# BRADBURY'S ELEMENTARY GEOMETRY AND TRIGONOMETRY.

BY

WILLIAM F. BRADBURY, A. M.,

HOPKINS MASTER IN THE CAMBRIDGE HIGH SCHOOL; AUTHOR OF A TREATISE ON TRIGONOMETRY AND SURVEYING, OF AN ELEMENTARY GEOMETRY AND TRIGONOMETRY, AND OF AN ELEMENTARY ALGEBRA.

BOSTON:
THOMPSON, BIGELOW, AND BROWN,
25 & 29 CORNHILL.
1872.

Educ T 148.72.236

# EATON'S

# Series of Mathematics,

**USED WITH UNEXAMPLED SUCCESS IN THE BEST SCHOOLS AND ACADEMIES OF THE COUNTRY.**

EATON'S PRIMARY ARITHMETIC.
EATON'S INTELLECTUAL ARITHMETIC.
EATON'S COMMON SCHOOL ARITHMETIC.
EATON'S HIGH SCHOOL ARITHMETIC.

EATON'S ELEMENTS OF ARITHMETIC.
EATON'S GRAMMAR SCHOOL ARITHMETIC.

BRADBURY'S EATON'S ELEMENTARY ALGEBRA.
BRADBURY'S ELEMENTARY GEOMETRY.
BRADBURY'S ELEMENTARY TRIGONOMETRY.
BRADBURY'S GEOMETRY AND TRIGONOMETRY, in one volume.
BRADBURY'S TRIGONOMETRY AND SURVEYING.

KEYS OF SOLUTIONS TO COMMON SCHOOL AND HIGH SCHOOL ARITHMETICS, TO ELEMENTARY ALGEBRA, GEOMETRY, AND TRIGONOMETRY, AND TRIGONOMETRY AND SURVEYING, *for the use of Teachers.*

Entered according to Act of Congress, in the year 1872,
BY WILLIAM F. BRADBURY,
in the Office of the Librarian of Congress, at Washington.

UNIVERSITY PRESS: WELCH, BIGELOW, & CO.,
CAMBRIDGE.

# KEY

TO

# BRADBURY'S ELEMENTARY GEOMETRY.

---

## BOOK I.

### PRACTICAL QUESTIONS.

1. 1st. They do, unless parallel (1). 2d. They do (1).

2. It does not.

3. Complements, $63^\circ$, $39^\circ$, $-1^\circ$, $-63$ (5).
Supplements, $167^\circ$, $97^\circ$, $83^\circ$, $-37^\circ$ (6).

4. $180^\circ - (15^\circ + 27^\circ + 99^\circ) = 39^\circ$ (7).

5. $360^\circ - (11^\circ + 53^\circ + 74^\circ + 19^\circ + 117^\circ) = 86^\circ$ (9).

6. Perpendicular (2).

7, 8. Yes (22), (23). (See Fig. II. 30.)

9. $180^\circ - (32^\circ + 43^\circ) = 105^\circ$ (34).

10. $90^\circ - 24^\circ = 66^\circ$ (37).

11. $180^\circ \div 3 = 60^\circ$ (38).

12. $(180^\circ - 14^\circ) \div 2 = 83^\circ$.

13. $90^\circ \div 2 = 45^\circ$.

14. $180^\circ \div 5 = 36^\circ$, angle at vertex.
$(180^\circ - 36^\circ) \div 2 = 72^\circ$, angle at base.

15. $180^\circ \div 4 = 45^\circ$, angle at base.
$45^\circ \times 2 = 90^\circ$, angle at vertex.

16. Not necessarily. (See Fig., II. 22.)

17. 1st. Yes. 2d. Not necessarily.

18. 1st. Yes. 2d. Not necessarily.

19. 1st. Three; the two not parallel and one of the parallel sides. 2d. Three.

20. $180°(5 - 2) \div 5 = 108°$; $180°(6 - 2) \div 6 = 120°$; $180°(8 - 2) \div 8 = 135°$; $180°(10 - 2) \div 10 = 144°$; $180°(12 - 2) \div 12 = 150°$ (67).

21. $(8 + 13) \div 2 = 10.5$ (66).

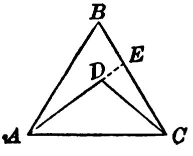
22. Opposite angle 120° (62); each of the other angles $180° - 120° = 60°$.

## EXERCISES.

**68.** (Fig. Art. 12.) As $ABC$ and $DEG$. Produce $DE$. $ABC = GEH$ (12); $DEG$ is supplement of $GEH$, and therefore also of $ABC$.

**69.** (Fig. Art. 26.) $AB + BC > AC$ (*Axiom* 9)

$$\therefore AB > AC - BC$$



**70.** Produce $AD$ till it cuts $BC$ at $E$. $AB + BE > AE$ and $DE + EC > DC$ (*Axiom* 9); adding these two inequalities, we have

$$AB + BE + DE + EC > AE + DC$$

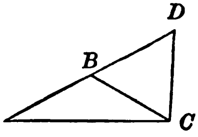
but $BE + EC = BC$ and $AE = AD + DE$

$\therefore$ subtracting $DE$ from both sides,

$$AB + BC > AD + DC$$

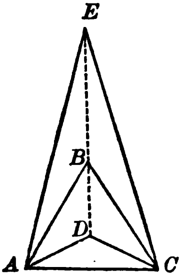
**71.** (Fig. 70.) $ADC > DEC$; but $DEC > ABE$ (39); much more then is $ADC > B$.

**72.** Let $ABC$ be the given isosceles triangle.



$$\text{Angle } BCA = 180° \div 6 = 30°$$
$$\text{Angle } ABC = 120°$$
$$\therefore \text{Angle } DCB = 90° - 30° = 60°$$
$$\text{Angle } CBD = 180° - 120° = 60°$$
and $$\text{Angle } D = 180° - (60° + 60°) = 60°$$
and $BDC$ is equiangular and therefore equilateral (46).

**73.** Let $ABC$ be equilateral, $ADC$ and $AEC$ isosceles. As all the triangles are isosceles, a perpendicular bisecting $AC$ will pass through $D$, $B$, and $E$ (53); that is, $D$, $B$, and $E$ are in the same straight line. Draw $DE$; $DE$ bisects the angles at $E$ and $B$ (43).

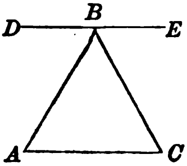


1st. $$\text{Angle } ABC = BCA = 60°$$
$$DBC = BCD = 30°$$
$$\therefore DCA = CAD = 60° - 30° = 30°$$
and $$ADC = 180° - (30° + 30°) = 120°$$
$$\therefore DCA = \tfrac{1}{4} ADC$$

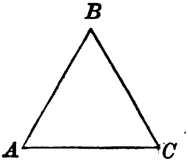
2d. As $DBC$ is an exterior angle of the triangle $BEC$, and $BEC$ is isosceles,

$$BEC = ECB = 15°$$
$$\therefore AEC = 30°$$
and $$ECA = 60° + 15° = 75°$$
$$\therefore ECA = \tfrac{5}{2} AEC$$

**74.** $DBA = BAC$ and $EBC = BCA$ (17). Hence,



$$DBA + ABC + EBC = A + ABC + C$$
but (7)
$$DBA + ABC + EBC = \text{two right angles}$$
$$\therefore A + ABC + C = \text{two right angles.}$$

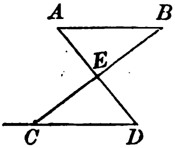


**75.** Stand at $A$ facing $B$; walk along $AB$ to $B$; at $B$ turn through the angle $B$ and your back will be toward $C$; move backward along $BC$ to $C$; at $C$ turn through the angle $C$ and your face will be towards $A$; move along $CA$ to $A$; at $A$ turn through the angle $A$ and your back will be toward $B$; now you are facing in a direction exactly the opposite to your first direction; that is, in turning through the angles of the triangles you have turned just half round, or through two right angles.

(Be careful to turn through the *interior* angles of the trianangles. If you turn through the exterior you will turn entirely round.)

**76.** For in both cases the two perpendiculars must be parallel (18); and if they coincide at any one point must coincide throughout and form one and the same straight line.

**77.** They are parallel because the internal angles on the same side are supplements (18).



**78.** Let $AD$, joining the parallels $AB$, $CD$, be bisected at $E$ by $BC$; then $BE = EC$. For the triangles $ABE$, $EDC$, having the side $AE = ED$, and the adjacent angles equal (17), (11), are equal (41); $\therefore BE = CE$.

**79.** The triangles $CBE$, $EBD$ are equal (40);

$$\therefore CE = ED$$

But (*Axiom* 9) $\quad AE + EC > AC$

$$\therefore AE + ED = AD > AC$$

**80.** If the included angle is not greater, it is either equal to it or less; it is not equal to it, for then the third sides would be equal (40), which is contrary to the hypothesis; it is not less, for then the third side of the first would be less than the third side of the other (79), which is also contrary to the hypothesis; therefore it is greater.

**81.** In the triangles $ABC$, $DEF$ (Fig. Art. 41), the angle $B = E$; for if $B > E$, $AC > DF$, or if $B < E$, $AC < DF$, which are both contrary to the hypothesis; $\therefore$ (40) triangle $ABC = DEF$. Or, as above, we can also prove two other angles equal, then the triangles will be equal by (41).

**82.** (Fig. Art. 62.) Draw $BD$. The two triangles $ABD$, $BCD$ are equal (48); therefore angle $ABD = BDC$ (49), and $AB$ is parallel to $DC$ (18); also angle $CBD = BDA$, and $BC$ is parallel to $AD$.

**83.** (Fig. Art. 59.) By hypothesis $Q = S$ and $R = T$;

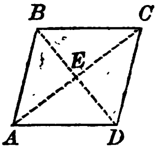
$$\therefore Q + R = S + T$$

But $$Q + R + S + T = 4 \text{ right angles } (67);$$

$$\therefore Q + R = 2 \text{ right angles};$$

therefore $QT$ is parallel to $RS$ (18). In like manner it can be proved that $RQ$ is parallel to $ST$.

**84.** It is not. Suppose the triangle $ABD$ (Fig. Art. 62), so situated that the side $AB$ shall be adjacent to $DC$, and $AD$ to $AB$; then, although the diagonal $BD$ would divide the quadrilateral into two equal triangles, the figure would not be a parallelogram.



**85.** Triangle $BCE = AED$, since $BC$ is equal to $AD$ (62), and the angles adjacent to $BC$, $AD$ are equal (17);

$$\therefore BE = ED, \text{ and } AE = EC.$$

**86.** (Fig. 85.) Angle $BEC = AED$ (11), and $BE$, $EC = ED$, $EA$, respectively; $\therefore$ $BC = AD$ (40); and as angle $CBD = BDA$, $BC$ is also parallel to $AD$;

$$\therefore ABCD \text{ is a parallelogram } (65).$$

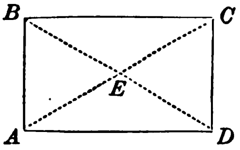
**87.** (Fig. 85.) Triangle $ABE = BCE$ (48);

$$\therefore \text{angle } AEB = BEC$$

and $BD$ is perpendicular to $AC$.

**88.** (Fig. 85.) If the diagonals bisect, the figure is a parallelogram (86). And the triangles $ABE$, $BCE$ have two sides $BE$, $EA = BE$, $EC$, and the included angles at $E$ equal; $\therefore AB = BC$; but $AB = CD$ and $BC = AD$;

$\therefore ABCD$ is a rhombus (60).

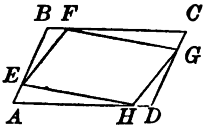


**89.** Right triangle $ACD = BCD$ (40); $\therefore AC = BD$.

**90.** (Fig. 85.) Right triangle $ABE = BCE$ (40, or 48); $\therefore$ angle $ABE = EBC$.

**91.** (Fig. 85.) Since angles $CBA + BAD =$ two right angles (17), $ABE + EAB =$ one right angle;

$\therefore BEA$ is a right angle (33).



**92.** Let $AE = BF = CG = DH$; $FGHE$ is a parallelogram. Since $EB$, $BF = GD$, $DH$, and angle $B = D$ (62), triangle $EBF = GDH$, and $EF = GH$; in like manner it can be proved that $FG = EH$; $\therefore FGHE$ is a parallelogram (82).

**93.** (Fig., II. 34.) The polygon $ABCDEF$ is divided into as many triangles as it has sides. The sum of the angles of all these triangles is equal to twice as many right angles as there are sides to the polygon; but the angles at $O$ are together equal to four right angles. The sum of the angles of the polygon is manifestly the same as the sum of the angles of the triangles minus the angles at $O$. That is, the sum of the angles of the polygon is equal to the sum of the angles of the triangles minus four right angles; or is equal to twice as many right angles as it has sides minus two.

**94.** The angle $ABC + b =$ two right angles (7), and so of he angles at $C$, $D$, $E$, &c.; $\therefore$ the sum of the angles of the polygon plus the angles $a$, $b$, $c$, $d$, $e$, $f$, is equal to twice as many right angles as the figure has sides; but the sum of the angles of the polygon is equal to twice as many right angles as the figure has sides, lacking four right angles;

$$\therefore a + b + c + d + f = \text{four right angles.}$$

**95.** The sum of all the angles formed in a plane about a point is equal to four right angles, or 360°. No regular polygon, therefore, can be taken unless the number of degrees in each angle is a divisor of 360.

Each angle of an equilateral triangle (the regular figure whose angles are least) is equal to 60°.

All the divisors of 360, not less than 60, are 60, 72, 90, 120, 180, and 360; of which only 60, 90, and 120 can be the number of degrees in an angle of a regular polygon. These numbers represent the number of degrees in each angle of an equilateral triangle, a square, and an equilateral hexagon respectively.

**96.** The sum of the angles at each point is equal to four right angles (9); and hence no polygons can be used unless their angles are such that a combination of two kinds of them can be made so that the sum of the angles at one point shall equal four right angles, or 360°. 1st. Three triangles and two squares $(3 \times 60^\circ + 2 \times 90^\circ = 360^\circ)$. 2d. Two triangles and two hexagons $(2 \times 60^\circ + 2 \times 120^\circ = 360^\circ)$. 3d. One triangle and two dodecagons $(60^\circ + 2 \times 150^\circ = 360^\circ)$. 4th. One square and two octagons $(90^\circ + 2 \times 135^\circ = 360^\circ)$. No other combination of two kinds of regular polygons can be taken so that the sum of the angles at one point will equal four right angles, or 360°.

**97.** No polygons can be used unless their angles are such that a combination of three kinds of them can be made so that the sum of the angles at one point shall equal four right angles, or 360°. 1st. One triangle, two squares, and one hexagon $(60^\circ + 2 \times 90^\circ + 120^\circ = 360^\circ)$. 2d. One square, one hexagon, and one dodecagon $(90^\circ + 120^\circ + 150^\circ = 360^\circ)$. No other combination of three kinds of regular polygons can be taken so that the sum of the angles at one point will equal four right angles, or 360°.

# BOOK II.

## PRACTICAL QUESTIONS.

1. Area $= 35 \times 25 = 875$ square inches.
Perimeter $= (25 + 35)2 = 120$ inches.

2. Area $= 20 \times 12 = 240$ square feet.

3. Area $= 14 \times 8 \div 2 = 56$ square feet.

4. Square surface $= 15\left(\frac{16 + 9}{12}\right) \div 2 = 15\frac{5}{8}$ square feet.

5. 3, 4, 5, and any multiples of them.

6. $\sqrt{50^2 - 40^2} = 30$ feet.

7. $\sqrt{67^2 - 40^2} = 53.75$ feet to the top of the ladder when it stands 40 feet from the wall.

$$\sqrt{67^2 - (53.75 + 10)^2} = 20.6 \text{ feet.}$$

8. $\sqrt{108^2 - 75^2} + \sqrt{108^2 - 45^2} = 77.7 + 98.2 = 175.9$ ft.

9. $1 : 9 = 20^2 : 60^2$. Ans. 60 rods.

10. $\sqrt{16^2 + 24^2} = 28.8$ feet.

11. $\sqrt{16^2 + 24^2 + 10^2} = 30.5$ feet.

12. $\sqrt{12^2 + 4^2 + 3^2} = 13$.

13. $\sqrt{12^2 - 6^2} = 10.4$ feet.

14. $t : T = 10^2 : 100^2 = 1^2 : 10^2 = 1 : 100$.

15. Surface of model : surface of church $= 10^2 : 120^2 = 1^2 : 12^2 = 1 : 144$. Ans. 144.

16. $\sqrt{125^2 - 75^2} \times 75 \div 2 = 100 \times 75 \div 2 = 3750$ sq. ft.

## EXERCISES.

**38.** On *A C* construct the square *D C*. Through *B* draw *B G* parallel to *C E*; take *A F* equal to *A B*, and through *F* draw *F H* parallel to *A C*. As *D I* and *I C* are parallelograms,

$DF = GI$ and $IH = BC$; but as $AD = AC$ and $AF = AB$, $FD = BC$; $\therefore$ $GI = IH$, and $GH$ is a square equal to the square described on $BC$. $FB$ is the square described on $AB$; and each of the rectangles $DI$, $IC$, is measured by $AB \times BC$.

$$\therefore DC = \overline{AC}^2 = \overline{AB}^2 + 2\,AB \times BC + \overline{BC}^2$$

**39.** If $AB = \frac{1}{2}AC$, $AB = BC = AF = FD = GI$, &c.; and the figures $DI$, $GH$, $IC$, $FB$ are squares equal to each other; $\therefore$ the square on $AC$ = four times the square on $AB$. Or, algebraically,

$$AC = 2\,AB;\ \therefore \overline{AC}^2 = 4\,\overline{AB}^2$$

**40.** On $AB$ construct the square $DB$. Through $C$ draw $CG$ parallel to $BE$; take $AF = AC$, and through $F$ draw $FH$ parallel to $DE$. On $DF$ construct the square $KF$. As $AB = AD$, and $AC = AF$, $BC = FD = KD = GE$. Hence, also, $KG = DE$, and each of the rectangles $KI$, $GB$ is measured by $AB \times BC$, and $KF = \overline{BC}^2$; and the whole figure or

$$\overline{AC}^2 + 2\,AB \times BC = \overline{AB}^2 + \overline{BC}^2$$

or

$$\overline{AC}^2 = \overline{AB}^2 - 2\,AB \times BC + \overline{BC}^2$$

**41.** On $AB$ construct the square $EB$. Through $C$ draw $CG$ parallel to $BF$; take $AK = AC$, and through $K$ draw $KH$ parallel to $EF$. $GH = \overline{BC}^2$. Produce $AB$ to $D$, making $BD = BC$, and complete the rectangle $HD$. As $AB = AE$ and $AC = AK$, $KE = BC = BD$, and the rectangle $EI = HD$; the rectangle $KD = AD \times DL = (AB + BC) \times (AB - BC)$, that is, the rectangle contained by the sum and difference of $AB$, $BC$; and the rectangle

$KD$ = the figure $AEGIHB = EB - GH = \overline{AB}^2 - \overline{BC}^2$

**42.** Letting $B$ and $A$ represent the base and altitude of a parallelogram $P$, $b$ and $a$ the base and altitude of a parallelogram $p$, we have (10)

$$P = B \times A, \text{ and } p = b \times a$$

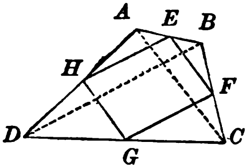
$$\therefore P : p = B \times A : b \times a$$

**43.** If $B = b$ in $P : p = B \times A : b \times a$
then (Pn. 21) $\qquad P : p = A : a$
or, if $A = a$, $\qquad P : p = B : b$

**44.** To the middle of the base; for the triangles thus formed are equal since they have the same altitude and equal bases.

**45.** (Fig., VI. 22.) Let $AHE$ be the triangle, and $GD$, $BF$ be parallel to $HE$. As $GD$, $BF$ are parallel, the triangles $AGD$, $ABF$, being mutually equiangular, are similar (20);

$$\therefore AG : GB = AD : DF$$



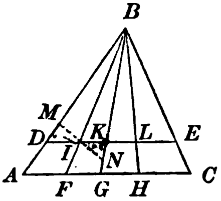
**46.** Let $E$, $F$, $G$, $H$ be the middle points of the sides of the quadrilateral $ABCD$; $EFGH$, formed by joining these points, is a parallelogram. Draw the diagonals $AC$, $BD$.

1st. As $E$ and $F$ are the middle points of $AB$, $BC$, two sides of the triangle $ABC$, $EF$ is parallel to $AC$ (18); in like manner, as $H$ and $G$ are the middle points of $AD$, $DC$, $HG$ is parallel to $AC$; hence (I. 15), $EF$ and $HG$ are parallel; for a like reason $HE$ and $GF$, both being parallel to $DB$, are parallel to each other; hence $EFGH$ is a parallelogram.

2d. As $ABC$ and $EBF$ are similar triangles (20)

$$AB : EB = AC : EF$$

but as $AB = 2\,EB$, $AC = 2\,EF = EF + HG$.
In like manner it can be proved that $BD = EH + FG$;
hence $\qquad EF + FG + GH + HE = AC + BD$



**47.** As $DI$ is parallel to $AF$, the triangles $BID$, $BFA$, being mutually equiangular, are similar (20); also $BKI$ and $BGF$. Hence, $DI : AF = BI : BF$
and $\qquad IK : FG = BI : BF$
Hence (Pn. 11) $DI : AF = IK : FG$
In like manner it can be shown that

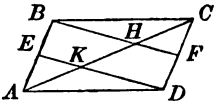
$$IK : FG = KL : GH = LE : HC$$

**48.** A line divided in the same proportion as a side of a triangle by lines drawn from the vertex to the side is parallel to this side. If $DE$ (Fig., 47), divided in the same proportion as $AC$ by the lines $BF$, $BG$, $BH$, is not parallel to $AC$, through $I$ draw $MN$ parallel to $AC$.

Then (47) $MI : IN = AF : FG$
But by hypothesis $DI : IK = AF : FG$
Hence (Pn. 11) $MI : IN = DI : IK$

and the two triangles $DMI$, $IKN$ are similar (23), and the angle $IDM = IKN$. Hence (I. 18) $BA$ is parallel to $BG$, which is absurd since these lines meet in $B$; therefore $DE$ is parallel to $AC$.

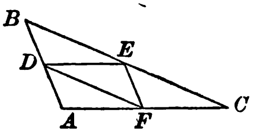
**49.** $BE$, $FD$ being equal and parallel, $BFDE$ is a parallelogram (I. 65), and $BF$ is parallel to $ED$. Now in the triangle $ABH$, as $EK$ is parallel to $BH$ and $AE = EB$, $AK = KH$; in like manner in the triangle $CDK$, as $HF$ is parallel to $KD$ and $CF = FD$, $CH = HK$;



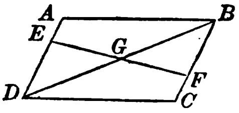
$$\therefore AK = KH = HC$$

**50.** (Fig., I. 47.) Let the two triangles $ABD$, $DBC$ have the sides $DB$, $DA$ of the one equal respectively to $DB$, $DC$ of the other, and the included angle $ADB$ be the supplement of $BDC$. If the triangles are placed as in the figure, as the angles $ADB$, $BDC$ are supplements of each other, $AD$ and $DC$ will form one and the same straight line (I. 10); and the triangles having a common vertex $B$, and their bases in the same straight line, have the same altitude; but base $AD = DC$. Hence (12) triangle $ABD = DBC$.

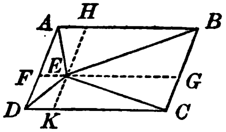
**51.** (Fig., I. 85.) Triangle $BCE = AED$ (I. 41); for $BC = AD$ (I. 62), and the angles adjacent $BC$ equal the angles adjacent $DA$ (I. 17); in like manner triangle $ABE = ECD$; and the two triangles standing on opposite sides are equal. Triangle $ABE$ is equivalent to $BCE$, for they have equal bases $AE$, $CE$ (I. 85), and the same altitude. Therefore $ABE$, $BCE$, $ECD$, $EDA$ are equivalent to one another.



**52.** As $D$, $E$, $F$ are the middle points of $AB$, $BC$, $CA$ respectively, the lines $FE$, $DF$, $ED$ are parallel respectively to $AB$, $BC$, $CA$ (18); hence $DBEF$, $DECF$, $ADEF$ are parallelograms; and these parallelograms are respectively bisected by their diagonals $DE$, $EF$, $DF$. Hence the four triangles are equal; or $DEF = \frac{1}{4} ABC$.



**53.** Let $EF$ pass through $G$, the middle of $BD$. The triangles $EGD$, $BGF$, having $DG$, $GB$, and the angles adjacent equal (I. 11; I. 17), are equal; hence $ED = BF$; and as $AD = BC$, $AE = FC$. Then the two polygons $ABFE$, $EFCD$ are equal (6).



**54.** Through $E$ draw $FG$, $HK$ parallel to $AB$, $AD$, respectively; the figures $FK$, $FH$, $HG$, $GK$, are parallelograms bisected by their diagonals.

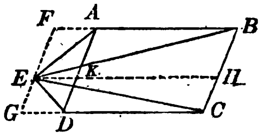
Therefore $FED + AEF = EKD + AHE$

or $AED = \frac{1}{2} AHKD$

In like manner $BEC = \frac{1}{2} HBCK$

hence $AED + BEC = \frac{1}{2} ABCD$

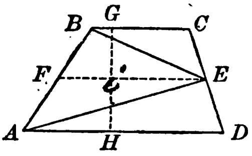
**55.** If a point without a parallelogram is joined to the vertices, the difference of the triangles formed by the joining lines and two opposite sides is equivalent to half the parallelogram.



Let $E$ be the point without the parallelogram $AC$. Through $E$ draw $EH$, $FG$ parallel to $AB$, $AD$, respectively; produce $BA$, $CD$ to meet $FG$. As $FH$ and $FK$ are parallelograms, the triangle $FBE = EBH$, and $FAE = EAK$;

hence $FBE - FAE = EBH - EAK$
but (11) $FBE - FAE = EAB = \frac{1}{2} ABHK$
hence $EBH - EAK = \frac{1}{2} ABHK$
In like manner $EHC - EKD = \frac{1}{2} KHCD$
Hence $EBC - EAD = \frac{1}{2} ABCD$

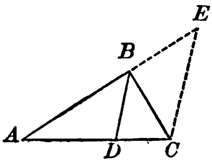
**56.** Let $GH$ be the altitude of the trapezoid. Draw $FE$ parallel to $BC$.
By (11) Area $BEF = \frac{1}{2} FE \times GI$
and Area $FEA = \frac{1}{2} FE \times IH$
Hence Area $BEA = \frac{1}{2} FE \times GH$
By (15) Area $BCD = FE \times GH$
Hence $BEA = \frac{1}{2} BCD$



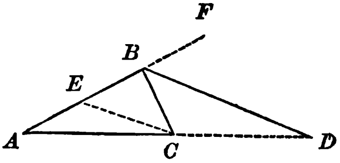
**57.** For in this case the triangles are mutually equiangular (I. 35), and therefore similar (20).

**58.** (Fig. 20.) On $AB$ cut off $AG = DE$ and draw $GH$ parallel to $BC$. Then (16)
$$AB : AG = AC : AH$$
or $AB : DE = AC : AH$
But by hypothesis $AB : DE = AC : DF$
Since the first three terms in the last two proportions are the same, their fourth terms must be equal; or $AH = DF$. In like manner it can be shown that $GH = EF$. Therefore the two triangles $AGH$, $DEF$ are equal (I. 48). But $AGH$ is similar to $ABC$; therefore $ABC$ is similar to $DEF$.

**60.** 1st. Let $B$, an angle of $ABC$, be bisected by $BD$. Through $C$ draw $CE$ parallel to $DB$, meeting $AB$ produced in $E$. As $BD$, $EC$ are parallel, angle $ABD = BEC$ and $DBC = BCE$ (I. 17); but $ABD = DBC$; hence $BCE = BEC$, and $BE = BC$. But as $BD$, $EC$ are parallel (16),



$$AB : BE = AD : DC$$
or, as $BE = BC$, $AB : BC = AD : DC$



2d. Let *CBF*, an exterior angle of *ABC*, be bisected by *BD*, which meets *AC* produced in *D*. Through *C* draw *CE* parallel to *DB*. As *BD*, *EC* are parallel, angle *CEB* $= DBF$ and $ECB = CBD$ (I. 17); but $CBD = DBF$; hence $CEB = ECB$, and $BE = BC$. But as *BD*, *EC* are parallel,

$$AB : BE = AD : DC$$

or, as $BE = BC$,

$$AB : BC = AD : DC$$

**61.** The two triangles *ABC*, *ABE*, having the same altitude, are as their bases (13); also the two triangles *ABE*, *ADE*, having the same altitude, are as their bases; that is,

$$ABC : ABE = AC : AE$$

and

$$ABE : ADE = AB : AD$$

By (Pn. 24)

$$ABC \times ABE : ABE \times ADE = AB \times AC : AD \times AE$$

or (Pn. 21)

$$ABC : ADE = AB \times AC : AD \times AE$$

**62.** Turn the triangle *ABF* on the point *A* in its own plane till *AB* coincides with *AG*, and as *AB* turns through a right angle, *AF* must turn through a right angle; therefore *F* will fall on *G*, and the triangle *ABF* coincide with *AGC*; hence $AGC = ABF$. But $AGC = \frac{1}{2} AH$ (11), and $ABF = \frac{1}{2} AL$; hence $AH = AL$. In like manner, by joining *AI* and *BE*, *LC* can be proved equivalent to *CI*.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

**63.** First, *GH*, *KI*, produced, will meet; for if they do not, they must be parallel; but *KI* is parallel to *BH*, and cannot be parallel to *GH*, a side of the same square *AH* adjacent to *BH*. Let them meet in *M* (write *M* where they meet, in Fig., 62); join *BM*; we are to prove that *LB*, *BM* form a straight line. The triangle $BMI = ABC$ (I. 40), for $BI = BC$, $MI = HB = AB$, and the angle $MIB = ABC$, both being right angles; therefore angle $IBM = BCA$. But

$DBC + BCD =$ a right angle (I. 37); therefore $DBC + IBM =$ a right angle; and $DBC + CBI + IBM = DBI + IBM =$ two right angles; therefore $LB, BM$ form a straight line (I. 10).

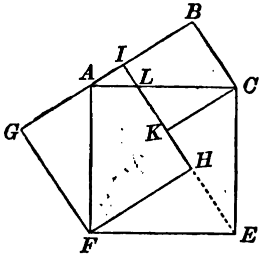
(Or, having proved above the angle $IBM = BCA$, as $BCA = ABD$ (24), then $IBM = ABD$; but the vertical angles made by $AI$ and $LB$ produced, would be equal; therefore, if $LB$ is produced, it must coincide with $BM$.)

*Corollary.* The parallelogram formed by producing $GH$, $KI$ is double the triangle $ABC$.

**64.** (In Fig., 27 write $N$ where $FA$, produced, meets $GH$, and $M$ where $GH$, $KI$, $LB$, produced, meet.)

The right angle $GAB = NAC$; take away from each the common part $NAB$, and $GAN = BAC$; and the right angle $AGN = ABC$; and the side $AG = AB$; therefore the triangle $AGN = ABC$, and $AN = AC = AF$. Hence the two parallelograms $ANMB$, $ADLF$, having equal bases $AN$, $AF$, and the same altitude, are equivalent; but $ANMB$, $AGHB$, having the same base $AB$, and the same altitude, are equivalent; therefore $AGHB$ is equivalent to $ADLF$, or $\overline{AB}^2 = AL$. In like manner (producing $EC$ till it meets $KI$), it can be proved that $\overline{BC}^2 = DE$.

$$\therefore \overline{AB}^2 + \overline{BC}^2 = AL + DE = \overline{AC}^2$$



**65.** Having constructed the squares as directed,

1st. The extremities of the two sides $GF$, $AF$, of the squares $GH$, $AE$, will be at the same point $F$; for, if the points $G$ and $F$ are joined, the triangle $AGF = ABC$; for by construction $GA = IB = BC$ and $AF = AC$; and as $FAC$ is a right angle, $BAC$ is the complement of $GAC$; but $BAC$ is also the complement of $BCA$ (I. 37); therefore $GAF = BCA$; and the two triangles $GAF$, $ABC$ are equal (I. 40), and $GF = AB =$ a side of the square $GH$, and the angle $G$ is a right angle; therefore $GF$ is a side of the square $GH$.

2d. Join $HE$; then the triangle $FHE = ABC$; for $FH = GI = AB$, and $FE = AC$, and the angle $HFE = BAC$; hence $FHE$ is a right angle, and $IH$, $HE$ form a straight line.

Now the square $AE$ is covered by parts of the squares $GH$ and $IC$; it only remains to prove the rest of $AE$, viz. $FHE + EKC$, equivalent to that part of the squares $GH$, $BK$, without $ACEF$. The triangle $FGA = FHE$, as both have been proved equal to $ABC$; and the rest of the two squares $GH$, $BK$, that is, $AIL$, $IBCL$, make up the triangle $ABC$; but $EKC = ABC$ (I. 50). Therefore the whole square $AE$ is equivalent to the squares $GH$ and $BK$.

**66.** By (40) $\overline{DC}^2 = \overline{AD}^2 + \overline{AC}^2 - 2\,AC \times AD$

Adding $\overline{BD}^2$ to each member

$$\overline{BD}^2 + \overline{DC}^2 = \overline{BD}^2 + \overline{AD}^2 + \overline{AC}^2 - 2\,AC \times AD$$

But (27) $\overline{BD}^2 + \overline{DC}^2 = \overline{BC}^2$ and $\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2$

hence $\qquad \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AC \times AD$

**67.** By (38) $\overline{DC}^2 = \overline{AD}^2 + \overline{AC}^2 + 2\,AC \times AD$

Adding $\overline{BD}^2$ to each member

$$\overline{BD}^2 + \overline{DC}^2 = \overline{BD}^2 + \overline{AD}^2 + \overline{AC}^2 + 2\,AC \times AD$$

But (27) $\overline{BD}^2 + \overline{DC}^2 = \overline{BC}^2$ and $\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2$

hence $\qquad \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 + 2\,AC \times AD$

**68.** If $A$ becomes a right angle, the perpendicular $BD$ coincides with $BA$, and $AD$ becomes zero. Hence $2\,AC \times AD = 0$, and (66) and (67) reduce to

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$$

If $C$ becomes a right angle, the perpendicular $BD$ coincides with $BC$, and $AD$ becomes in (66) equal to $AC$; in (67), as the point $D$ passes $A$, $AD$ becomes zero, then negative, and when $BD$ coincides with $BC$, $AD$ becomes equal to $-AC$.

Hence, in (66) $2\,AC \times AD = 2\,\overline{AC}^2$
and in (67) $2\,AC \times (-AD) = -2\,\overline{AC}^2$
Hence (66) and (67) both reduce to

$$\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2$$

**69.** (Fig., I. 51.) Let $E$ be the middle point of $BC$, a side of the triangle $BAC$; draw $AD$ perpendicular to $BC$.
In triangle $EAC$ (66) $\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2\,EC \times ED$
In triangle $BAE$ (67) $\overline{AB}^2 = \overline{AE}^2 + \overline{BE}^2 + 2\,BE \times ED$
Hence, as $BE = EC$,

$$\overline{AC}^2 + \overline{AB}^2 = 2\,\overline{AE}^2 + 2\,\overline{BE}^2$$

**70.** (Fig., I. 85.) In triangle $ABC$ (69)

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{BE}^2 + 2\,\overline{AE}^2$$

In triangle $ACD$ (69) $\overline{CD}^2 + \overline{DA}^2 = 2\,\overline{ED}^2 + 2\,\overline{AE}^2$
Hence, as $BE = ED$,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\,\overline{BE}^2 + 4\,\overline{AE}^2$$

But (39) $4\,\overline{BE}^2 + 4\,\overline{AE}^2 = \overline{BD}^2 + \overline{AC}^2$
Hence $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{BD}^2 + \overline{AC}^2$

**71.** The triangles $HIB$, $ABC$ being right angled at $B$, and $BI = BC$ and $HB = AB$, by (I. 40) $HIB = ABC$.

As the four angles at $A$ = four right angles, of which $GAB + FAC$ = two right angles, $GAF + BAC$ = two right angles; that is, $GAF$, $BAC$ are supplements of each other. Then the two triangles $GAF$, $ABC$, having $GA = AB$, $AF = AC$, and the included angles $GAF$, $BAC$ supplements of each other, are equivalent (50). In like manner $CKE$ can be proved equivalent to $ABC$.

(The triangle $CKE$ can also be proved equivalent to $ABC$ by turning it on the point $C$ in its own plane till $CK$ coincides with $CB$; then $AC$, $CE$ will form a straight line; and the two triangles will be equivalent, as they have equal bases and the same altitude. In like manner $GAF$ can be proved equivalent, and $HIB$ equal, to $ABC$.)

**72.** (Fig., 27.) From (27) $\overline{AB}^2 = AC \times AD$

and $\overline{BC}^2 = AC \times DC$

Hence $\overline{AB}^2 : \overline{BC}^2 = AC \times AD : AC \times DC$

or (Pn. 21) $\overline{AB}^2 : \overline{BC}^2 = AD : DC$

**73.** (Fig., 27.) $\overline{AC}^2 = \overline{AC}^2$

and from (27) $\overline{AB}^2 = AC \times AD$

Hence $\overline{AC}^2 : \overline{AB}^2 = \overline{AC}^2 : AC \times AD$

or (Pn. 21) $\overline{AC}^2 : \overline{AB}^2 = AC : AD$

**74.** Let $S$ and $D$ represent, respectively, a side and diagonal of any square.

By (27) $S^2 + S^2 = 2\,S^2 = D^2$

Hence (Pn. 14) $S^2 : D^2 = 1 : 2$

or (Pn. 22) $S : D = 1 : \sqrt{2}$

(Or, draw the diagonals of the square, and through the four vertices draw lines parallel to the diagonals; these will form a square equal to the square on the diagonal, and there will be eight equal triangles, of which the original square will contain four.)

**75.** (Fig., 30.) As all the triangles of $AD$ are similar to the triangles of $GK$, and similarly situated, angles $B$ and $F$ are equal to $H$ and $M$, and all the angles at $A$, $C$, $D$, $E$ are equal respectively to the angles at $G$, $I$, $K$, $L$; hence $AD$ and $GK$ are equiangular.

In the triangles $ABC$, $GHI$ (19)

$BC : HI = AC : GI$

and in $ACD$, $GIK$, $AC : GI = CD : IK$

Hence (Pn. 11) $BC : HI = CD : IK$

In like manner it can be proved

$CD : IK = DE : KL = EF : LM = FA : MG = AB : GH$

Hence (19) $AD$, $GK$ are similar polygons.

# BOOK III.

## PRACTICAL QUESTIONS.

1. $C = 3.14159 \times 10 \times 2 = 62.8318$ feet.

2. $D = 57 \div 3.14159 = 18.14367$ rods.

3. $A = 3.14159 \times 40^2 = 5026.544$ square feet.

4. $A = \frac{18}{3.14159} \times \frac{1}{2} \times \frac{18}{2} = 25.78312$ square inches.

5. $A = \frac{1}{2} C \times \frac{1}{2} D = \frac{1}{4} C \times D$. But $D = \frac{C}{\pi}$; $\therefore A = \frac{C^2}{4\pi}$, or $C = \sqrt{4 \pi A} = 2 \sqrt{116 \times 3.14159} = 38.1798$ feet.

6. $A = 3.14159 \times 54^2 - 3.14159 \times 40^2$
$= 3.14159 \times 1316 = 4134.33244$ square feet.

7. $A\text{'s} : B\text{'s} = \frac{3.14159}{4} \times 95^2 : 750 = 56.7056995 : 6$

8. $P$ of square — $P$ of circle
$= 4 \sqrt{A} - \sqrt{4 \pi A} = 4 \sqrt{160} - 2 \sqrt{160 \times 3.14159}$
$= 16 \sqrt{10} - 8 \sqrt{31.4159} = 50.5964 - 44.8399 = 5.7565$ rods.

9. $A$ of circ. $= 1 = \frac{1}{4} \pi D^2$; $\therefore D = 2 \sqrt{\frac{1}{\pi}} =$ diagonal of inscribed square. By (II. 74) $A$ of square $= \frac{1}{2}$ diagonal$^2$ $= \frac{1}{2}\left(2\sqrt{\frac{1}{\pi}}\right)^2 = \frac{2}{\pi} = .63662$ of a square metre.

10. $A$ of circ. $= 567 = \pi R^2$; $\therefore R$, or side of hexagon $= \sqrt{\frac{567}{\pi}} = 9\sqrt{\frac{7}{\pi}}$. Draw lines from the centre of the hexagon to its vertices, and the hexagon will be divided into six equal equilateral triangles whose altitude $= \sqrt{\frac{567}{\pi} - \frac{567}{4\pi}} = \sqrt{\frac{1701}{4\pi}} = \frac{9}{2}\sqrt{\frac{21}{\pi}}$. Area of the six triangles = area of the hexagon $= \frac{1}{2} \times 9\sqrt{\frac{7}{\pi}} \times \frac{9}{2}\sqrt{\frac{21}{\pi}} \times 6 = \frac{243}{2}\sqrt{\frac{7 \times 21}{\pi^2}} = \frac{1701}{2\pi}\sqrt{3} = \frac{1701}{6.28318} \times 1.73205 = 468.905$ square feet.

11. The areas of horizontal sections of the ropes must be as 1000 : 4000. Letting $a$, $A$ represent respectively the areas of the sections, $d$, $D$ the diameter,

$$a : A = d^2 : D^2 = 1 : D^2 = 1000 : 4000 = 1 : 4$$

Hence (Pn. 22) $1 : D = 1 : 2$, or $D = 2$ inches.

12. The quantity of water, regardless of friction, passing through a pipe varies as the area of a cross section of the pipe. Hence $5^2 : 1^2 = 25$ : time required, or 1 minute.

13. $\frac{9}{9}$ flow out while $\frac{1}{9}$ flows in; the cistern therefore is $\frac{8}{9}$ emptied in one hour.

Hence $\frac{8}{9} : 1 = 60$ : time required, or $67\frac{1}{2}$ minutes.

14. $3^2 - (1^2 + 1^2 + 1^2) : 3^2 = 3$ : time required;

or $6 : 9 = 3$ : time required, or $4\frac{1}{2}$ hours.

## EXERCISES.

**40.** Turn on the diameter one semicircle over upon the other, and they will coincide; otherwise there would be points in the circumference unequally distant from the centre.

**41.** For if it could meet the circumference in more than two points, there would be more than two equal straight lines (viz. the radii of the circle to the intersecting points) drawn from a point to a straight line, which is impossible (I. 51).

**42.** (Fig., 43.) Let $AC$ be the diameter, and $AB$ any chord. Draw $BC$; then $B$ is a right angle (23), and therefore (I. 33) greater than any other angle of $ABC$; hence (I. 47) $AC > AB$.

**43.** Place the arcs as in the Fig., and join $AC$. If the arc $AB > BC$, the angle $C$, which is measured by half of the arc $AB$ (21), must be greater than $A$ which is measured by half $BC$; therefore (I. 47) the chord $AB > BC$.

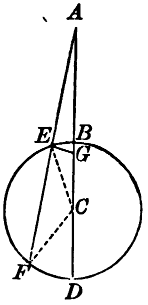
*Conversely.* If the chord $AB > BC$, the angle $C > A$ (I. 47); hence the arc $AB$, which is double the measure of the angle $C$, must be greater than $BC$, which is double the measure of $A$.

When the sum of the arcs is greater than a circumference the greater arc is subtended by the less chord. For if the arc $BDA < CDB$, the whole circumference — $BDA >$ the whole circumference — $CDB$, or the arc $AB > BC$. But it has just been proved that when the arc $AB > BC$, the chord $AB > BC$; therefore when the arc $BDA < CDB$, the chord $AB > BC$.

This proposition can also be proved by joining the centre and the extremities of the arcs, then referring to (I. 79; I. 80).

**44.** If we join the centre of the circle and the extremities of the chords, we shall form equal triangles (I. 48); therefore their corresponding altitudes, or their distances from the centre, must be equal. If (Fig., 21, 3d), $BL > BA$, $BL$ is nearer the centre. By the previous part of this proposition $BA$ is the same distance from the centre wherever it is placed. Place the two chords as in the figure. As the chord $BL > BA$, the arc $BL > BA$ (43); therefore $BL$ must fall between $BA$ and the centre, that is, is nearer the centre than $BA$.

**45.** 1st. When the point is without the circle. Let $A$ be the point, $AD$ a secant passing through the centre, $AF$ any other secant of the circle from $A$; $AB < AE$, and $AD > AF$.



Draw the radii $CE$, $CF$;

$$AC < AE + EC$$

Taking from each member $BC = EC$

$$AB < AE$$

And $AC + CF > AF$

Hence, as $CF = CD$, $AD > AF$

2d. When the point is within the circle. Let $G$ be the point, $BGD$ be a diameter, and $GE$ any other line from $G$ to the circumference, $GB < GE$ and $GD > GE$.

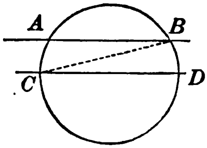
Draw the radius $CE$; $EC = BC < GE + GC$

Taking from each member $GC$, $GB < GE$

And $GC + CE > GE$

Hence, as $CE = CD$, $GD > GE$

2



**46.** Let $AB$, $CD$ be parallels intersecting the circumference. Draw $BC$.
By (I. 17) angle $ABD = BCD$
Hence (21) arc $AC = BD$

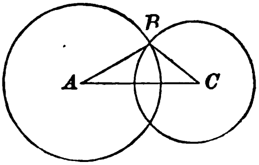
**47.** $CD$, being perpendicular to $DB$, is shorter than any oblique line $CB$ (I. 51); that is, $CB$ is longer than the radius, and the point $B$ must be without the circumference; that is, $DB$ cannot cut the circumference, and is therefore a tangent (9).

**48.** (Fig., 46.) Join $A$ to $D$. If $AD$, $BC$ are diameters, $AB$, $CD$ are parallel. For (11) the arc $AC = BD$.
Hence angle $ABC = BCD$, and $AB$, $CD$ are parallel (I. 18).

**49.** (Fig., 57.) The angles at $E$ are equal (I. 11), and angle $A = B$ (22); therefore the triangles $AED$, $BEC$ are equiangular (I. 35), and similar (II. 20).

**50.** For if a perpendicular is drawn to the middle of this chord, it must pass through the centres of both circles (16).

**51.** For if a perpendicular is drawn to the tangent at the point of contact, it must pass through the centres of both circles (47).



**52.** For (I. 69) $AC < AB + BC$
and $AC \ngtr AB - BC$

**53.** For every angle inscribed in a segment greater than a semicircle is measured by half of an arc which is less than a semi-circumference, or by less than a quadrant (21); that is, is less than a right angle, or acute. And every angle inscribed in a segment less than a semicircle is measured by one half of an arc which is greater than a semi-circumference, or by more than a quadrant; that is, is greater than a right angle, or obtuse.

**54.** The right angle $ABD$ is measured by half the arc $ACB$, and $ABC$ by half $AC$ (21); therefore $ABD - ABC$

$= CBD$ is measured by half the arc $ACB$ minus half $AC$, that is, by half $CB$.

**55.** Angle $B$ is measured by half $CD$, and $C$ by half $AB$ (21); therefore $B + C$ is measured by half $(CD + AB)$; but (I. 39) $BEA = B + C$; hence $BEA$ is measured by half $(CD + AB)$.

**56.** If $E$, the point of intersection, is moved;

1st. When $E$ is at the centre, $CD = AB$, and one half of $(CD + AB) = CD$, or $AB$; that is, an angle at the centre is measured by the arc included between its sides, as stated in (14).

2d. When $E$ is on the circumference, one of the arcs is zero, and hence the angle made by the two chords is measured by half the remaining arc which is included by the chords, as stated in (21).

**57.** The triangles $AED$, $BCE$ are similar (49);

$$\therefore AE : BE = ED : EC$$

**58.** For the sum of the arcs included by two opposite angles would be the whole circumference; that is, the angles are together measured by half a circumference, that is, by a semicircumference; hence they are together equal to two right angles, or are supplements of each other (I. 6).

**59.** (Fig., VI. 45.) A circumference can be made to pass through $A$, $D$, and $C$; for, as $AD$, $DC$, being adjacent sides of a quadrilateral, can neither be parallel nor in the same straight line, there must be a point where the perpendiculars bisecting $AD$ and $CD$ will meet; and the point of intersection is equally distant from $A$, $D$, and $C$ (I. 53). Draw the chord $AC$; $B$, being the supplement of $D$, is equal to any angle in the segment $ABC$ (58); but if the vertex $B$ is without the arc $ABC$, the angle $B$ is less, if within greater, than any angle whose vertex is in the arc (I. 71); hence $B$ is in the arc, and the vertex of any angle, not the supplement of $D$, cannot be in the arc $ABD$. Further, if $B$ is not the supplement of $D$, it cannot fall in the circumference passing through $C$, $D$, and $A$ (58); that is, a quadrilateral whose opposite angles are not supplementary cannot be inscribed in a circle.

**60.** If $A$, $R$, $D$, $C$ represent, respectively, the area, radius, diameter, and circumference of a circle, and $a$, $r$, $d$, $c$ the same of any other circle.

By (32) $$A = \pi R^2 = \tfrac{1}{4}\pi D^2 = \frac{C^2}{4\pi}$$

and $$a = \pi r^2 = \tfrac{1}{4}\pi d^2 = \frac{c^2}{4\pi}$$

Hence $$A : a = \pi R^2 : \pi r^2 = \tfrac{1}{4}\pi D^2 : \tfrac{1}{4}\pi d^2 = \frac{C^2}{4\pi} : \frac{c^2}{4\pi}$$

or (Pn. 21) $$A : a = R^2 : r^2 = D^2 : d^2 = C^2 : c^2$$

**61.** (Fig., 33.) If in the sector $ABC$ the arc $AB$ is divided into an infinite number of smaller arcs, and their chords drawn, and lines from $C$ to the extremities of these chords, an infinite number of triangles will be formed whose bases will coincide with the arc $AB$ (25), whose altitude will be the radius of the circle, and whose area will be half the product of the sum of their bases by their altitude, or half the arc $AB$ by the radius of the circle. But the sum of the triangles makes up the sector; therefore the area of the sector is half the product of its arc by the radius of the circle.

**62.** (Fig., 16.) Segment $ABE$ = sector $ACB$ — triangle $ACB$.

Area of sector $ACB = \frac{1}{2}$ arc $AB \times CE$ (61)

Area of triangle $ACB = \frac{1}{2}$ base $AB \times CD$ (II. 11)

Hence area of segment $= \frac{1}{2}$ arc $AB \times CE - \frac{1}{2}$ base $AB \times CD$

**63.** If the vertices of the polygon are joined to the centre of the circle, the polygon will be divided into triangles whose altitude will be the radius of the circle; hence the area of these triangles, that is, of the polygon, will be half the product of the sum of the bases by the radius of the circle, that is, half the product of the perimeter by the radius of the circle.

**64.** The angle $A = CDB$ as both are measured by half the arc $CD$ (21; 54), and the angle $B$ is common to the two triangles $ABD$, $CBD$; therefore the triangles $ABD$, $CBD$ are similar (II. 57);

$$\therefore AB : BD = BD : BC$$

**65.** Angle $CFD = A + C$, or $A = CFD - C$

Angle $CFD$ is measured by $\frac{1}{2}\ CD$

Angle $C$ is measured by $\frac{1}{2}\ GF$

Hence angle $A$ is measured by $\frac{1}{2}\ CD - \frac{1}{2}\ GF = \frac{1}{2}(CD - GF)$

**66.** When $A$ reaches the circumference, the arc $GF$ becomes zero, and $\frac{1}{2}(CD - GF) = \frac{1}{2}\ CD$.

If $A$ moves away from the circumference, if the points $C$, $D$ remain fixed, the arc $GF$ will increase; and when $A$ is at an infinite distance from the circumference $AC$ and $AD$ will be parallel, and the angle at $A$ will be zero;

$$\therefore \tfrac{1}{2}(CD - GF) = 0, \text{ or } CD = GF$$

**67.** Angle $C = D$ (22), and $A$ is common to the two triangles $AFC$, $ADG$; hence $AFC$ is similar to $ADG$ (II. 57);

$$\therefore AC : AD = AF : AG$$

**68.** By (54) Angle $ABE = AEB$

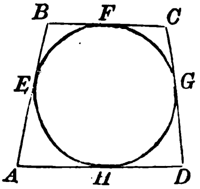
hence (I. 45) $AB = AE$

**69.** The triangle $ABC$ is right-angled at $B$ (23);

hence (II. 26), $AD : DB = DB : DC$

**70.** The triangle $ABC$ is right-angled at $B$ (23);

hence (II. 25) $AC : CB = CB : CD$



**71.** By (68)

$$AE = AH$$
$$BE = BF$$
$$CG = CF$$
$$DG = DH$$

$$\therefore AE + BE + CG + DG = AH + BF + CF + DH$$

or

$$AB + CD = AD + BC$$

# BOOK IV.

## EXERCISES.

**13.** (Fig., 4.) The plane *MN*, or any other plane, can take any one of the infinite number of positions *MN* will be in, as it turns on *AB* as an axis; that is, an infinite number of planes can pass through *AB*.

**14.** If there could be two, they would be in the same plane (4); and the intersection of this plane with the other would be a straight line (5); and then there would be two perpendiculars from a point to a straight line (the three lines in the same plane), which is impossible (I. 76).

**15.** For if it is not perpendicular, draw at this point a line that is perpendicular to the plane of the lines; then this will be perpendicular to each of the two intersecting lines (1); and there will be two perpendiculars from a point to a straight line (both perpendiculars and the straight line in the same plane), which is impossible (I. 76); therefore the first line must be perpendicular to the plane of the intersecting lines.

**16.** Parallel lines have the same direction (I. 13); they must therefore have the same difference of direction from the same plane, that is, be equally inclined to it.

**17.** Lines equally inclined to the same plane are parallel to one another. This is not true; they may or may not be parallel. *AC*, *AD*, and *AE* (Fig., 6) are equally inclined to the plane *MN*, but are not parallel.

**18.** (Fig., 11.) Let *AB* be parallel to *DE*; *AB* is parallel to the plane *PQ*. For if it is not parallel to the plane, it will, if produced, meet the plane. Through the point where it meets the plane draw a line *X*, in the plane *PQ*, parallel to *DE*; this line will also be parallel to *AB* (I. 15); and therefore two parallel lines $AB$ and $X$ meet, which is absurd (I. 14); therefore $AB$ cannot meet the plane $PQ$; that is, $AB$ is parallel to the plane $PQ$.

**19.** Lines parallel to a given plane are parallel to any line of the plane. This is not true; they may or may not be parallel. (Fig., 11) $AB$, parallel to the plane $PQ$, is not parallel to $FE$; but it is parallel to any line, as $DE$, that is at the same time in $PQ$ and in a plane including $AB$. For if $AB$ could meet the plane it would meet it in the line $DE$, or in some line parallel to $DE$; but it cannot meet the plane, therefore it cannot meet $DE$, and therefore, being in the same plane as $DE$, it is parallel to $DE$, or to any other line parallel to $DE$.

**20.** (Fig., 11.) Let $AD$ be a line meeting the two parallel planes $MN$, $PQ$. Let $AE$ be any plane including $AD$, whose intersections with $MN$, $PQ$ are $AB$, $DE$; $AB$, $DE$ are parallel (9); therefore $AB$, $DE$ differ equally in direction from $AD$; therefore the planes $MN$, $PQ$ differ equally in direction from $AD$, that is, are equally inclined to $AD$.

**21.** Planes equally inclined to the same straight line are parallel. It is not true; they may or may not be. The planes $AE$, $CE$ (Fig., 11), if the angle $A = C$, are equally inclined to the line $AC$, but they are not parallel.

**22.** (Fig., 11.) Let $AD$, $CF$ be two parallel lines included between the parallel planes $MN$, $PQ$. Let the plane of the parallels be $AF$, whose intersections with $MN$, $PQ$ are $AC$, $DF$. $AC$ is parallel to $DF$ (9); therefore $AD = CF$ (I. 64).

# BOOK V.

## PRACTICAL QUESTIONS.

1. By (14) Convex surface $= \frac{8}{12} \times 6 \times 2 = 8$ square feet.
By (II. 34 ; III. 33 ; II. 28)
Apothem of base $= \sqrt{8^2 - 4^2} = \sqrt{48} = 4\sqrt{3} = 6.9282$ inches.
By (II. 37) Area of base $= \frac{1}{2} \times 8 \times 6 \times 6.9282 \div 144$
$= 1.1547$ square feet.
$\therefore$ whole surface $= 8 + 1.1547 \times 2 = 10.3094$ square feet.

2. By (15) Convex surface $= 2\pi RA = 2 \times 3.14159 \times \frac{6}{12} \times 3$
$= 9.42477$ square feet.
By (III. 32) Area of base $= \pi R^2 = 3.14159 \times \frac{1}{4} = .7854$ sq. ft.
$\therefore$ whole surface $= 9.42477 + .7854 \times 2 = 10.99557$ sq ft.

3. From (11) and (II. 9) No. square feet $= 5^2 \times 6 = 150$
By (24) No. solid feet $= 5^3 = 125$

4. From (10) and (14)
No. square feet $= (4 + 7)2 \times 9 + 4 \times 7 \times 2 = 254$
By (21) No. solid feet $= 4 \times 7 \times 9 = 252$

5. By (25) $V =$ Base $\times$ Altitude $= 1.1547 \times 2$
$= 2.3094$ cubic feet.

6. By (26) $V = \pi R^2 A = 3.14159 \times \frac{1}{4} \times 3 = 2.3562$ cub. ft.

7. By (27) $V : v = A^3 : a^3$, or $3 : 1 = 729 : 243$
$\therefore a = \sqrt[3]{243} = 6.24$ feet.
$V ; v = P^3 : p^3$, or $3 : 1 = 216 : 72$
$\therefore p = \sqrt[3]{72} = 4.16$ feet.

8. By (27) $v : V = a^3 : A^3 = 3^3 : 6^3 = 1^3 : 2^3 = 1 : 8$

9. By (41) Convex surface $= 2 \times 8 \times \frac{1}{2} \times 3 = 24$ sq. ft.

10. By (42) Convex surface $= \pi R S = 3.14159 \times 2 \times 5$
$= 31.4159$ square feet.
By (III. 32) Area of base $= \pi R^2 = 3.14159 \times 2^2$
$= 12.56636$ square feet.
$\therefore$ whole surface $= 31.4159 + 12.56636 = 43.98226$ sq. ft.

11. By (51) $V = \frac{1}{3} \times 3^2 \times 10 = 30$ cubic feet.

12. (Fig., 33.) By (II. 28)
$A B =$ Altitude $= \sqrt{5^2 - 2^2} = \sqrt{21} = 4.58257$ feet.
By (52) $V = \frac{1}{3} \pi R^2 A = \frac{1}{3} \times 3.14159 \times 2^2 \times 4.58257$
$= 19.195$ cubic feet.

13. By (43) Convex surface $= (18 + 12)\frac{6}{2} = 90$ sq. ft.

14. The part necessary to complete the whole pyramid is a pyramid similar to the whole pyramid (36; 39; 4). Hence, if $S =$ the slant height of the whole pyramid, and $s =$ the slant height of the small pyramid,
by (4) $S : s = 18 : 12$
Hence (Pn. 18) $S - s : S = 6 : 18$
or $6 : S = 6 : 18$ Hence $S = 18$ feet.

15. $S = \sqrt{(11 - 5)^2 + 8^2} = \sqrt{100} = 10$ feet.
By (45) Convex surface $= (R + r)\pi S = 16 \times 3.14159 \times 10$
$= 502.6544$ sq. ft.
By (III. 32) Area of the two bases $= \pi R^2 + \pi r^2 = \pi(R^2 + r^2)$
$= 3.14159 \times 146 = 458.67214$ sq. ft.
$\therefore$ whole surface $= 502.6544 + 458.67214 = 961.32654$ sq. ft.

16. By (51) $V = \frac{1}{3} \times 8 \times 6 \times 25 = 400$ cubic feet.

17. By (52) $V = \frac{1}{3} \pi R^2 A = \frac{1}{3} \times 3.14159 \times 5^2 \times 20$
$= 523.598$ cubic feet.

18. The part cut off is a cone similar to the whole cone (36; 39; 4). Hence, if $V =$ the volume of the whole cone, $v =$ the volume of the small cone, then $V - v =$ the volume of the frustum.

By (53) $V : v = 20^3 : (20 - 5)^3 = 20^3 : 15^3 = 4^3 : 3^3 = 64 : 27$

Hence (Pn. 18) $V - v : V = 64 - 27 : 64$

or $V - v : 523.598 = 37 : 64$

and $V - v = 523.598 \times 37 \div 64 = 302.705$

19. The part cut off is a cone similar to the whole cone (36; 39; 4). Hence, if $V =$ the volume, $A =$ the altitude of the whole cone, and $v =$ the volume, $a =$ the altitude of the cone cut off, $A - a =$ the distance required, by (27)

$V : v = A^3 : a^3$, or $1 : \frac{1}{2} = 12^3 : a^3$, or $2 : 1 = 1728 : a^3$

Hence $a^3 = 864$, or $a = 9.524$

and $A - a = 12 - 9.524 = 2.476$ feet

20. By (62) $S = 4\pi R^2 = 4 \times 3.14159 \times 6^2$

$= 452.38896$ square feet.

21. By (64) $V = \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.14159 \times 8^3$

$= 2144.65877$ cubic feet.

22. By (65) $v : V = r^3 : R^3 = 4^3 : 8^3 = 1^3 : 2^3 = 1 : 8$

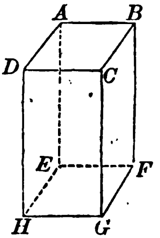
23. Spheres are always similar. For if any radius varies in length, all the radii must vary (54); and the diameter and the circumference, or any homologous lines, vary as the radius (58; 59; III. 28). Cones are not always similar. A cone may increase in altitude without changing its base.

24. Four. The least number of planes that can form a solid angle, as $G$ (Fig., 48), is three; and one more at least is necessary for a base, as $HIK$.

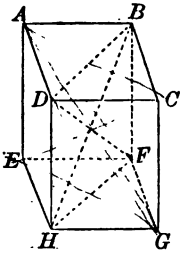
## EXERCISES.

**66.** If $S$, $A$, and $P$ represent, respectively, the convex surface, the altitude, and perimeter of any prism, or pyramid, and $s$, $a$, and $p$, the convex surface, altitude, and perimeter of any other pyramid,

by (14) $S = AP$, and $s = ap$
Hence $S : s = AP : ap$
If $A = a$ (Pn. 21), $S : s = P : p$



**67.** If $AG$ is a parallelopiped, $AF$, $BG$, $CH$, $DE$ are parallelograms (9). Hence $AB$, $BC$, $CD$, $DA$ are respectively equal and parallel to $EF$, $FG$, $GH$, $HE$ (I. 62); and the angles $A$, $B$, $C$, $D$ of $AC$ are respectively equal to $E$, $F$, $G$, $H$ of $EG$ (IV. 11); therefore $AC = EG$ (II. 6), and $AC$ is parallel to $EG$ (IV. 11).



**68.** Draw $DB$, $HF$, $DF$, $BH$. $BF$ and $DH$, each being parallel and equal to $CG$ (I. 62), are parallel and equal to each other; therefore $DBFH$ is a parallelogram (I. 65); and its diagonals $DF$, $BH$ bisect each other (I. 85). By drawing $AF$, $DG$, and $AG$ it can in like manner be shown that $AG$, $DF$, bisect each other; by drawing $AC$, $EG$, and $EC$ that $AG$, $EC$ also bisect each other. Hence $DF$, $BH$, $AG$, $EC$ bisect each other.

**69.** (Fig., 68.) Let $DF$ be a plane passing through the opposite edges, $DH$, $BF$, of the parallelopiped $AG$. The two prisms $ABD-EFH$, $BCD-FGH$, having the same altitude and equal bases (I. 63) are equal (19).

**70.** (Fig., 68.) Constructing as in (68),

1st. $DBFH$ is a rectangle, and $DF = BH$ (I. 89); also $AG = DF$; and $AG = EC$.

Therefore $DF = BH = AG = EC$

2d. $DBH$ is a right triangle;
hence (II. 27) $\overline{BH}^2 = \overline{DB}^2 + \overline{DH}^2$
But as $ABD$ is a right triangle,
$$\overline{DB}^2 = \overline{AB}^2 + \overline{AD}^2$$
Hence $\overline{BH}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{DH}^2$
or, as $AE = DH$, $\overline{BH}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{AE}^2$

**71.** If $AG$ is a cube, $AB = AD = AE$ (11);
hence $\overline{BH}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{AE}^2 = 3\,\overline{AB}^2$

**72.** If $V$, $B$, and $A$ represent, respectively, the volume, base, and altitude of any prism, and $v$, $b$, and $a$ the volume, base, and altitude of any other prism,
by (25) $V = B \times A$, and $v = b \times a$
Hence $V : v = B \times A : b \times a$

**73.** If $B = b$, the proportion in (72) becomes (Pn. 21)
$$V : v = A : a$$
or, if $A = a$, $V : v = B : b$

**74.** (Fig., 39.) The polygons $GI$, $BD$ are similar (39);
hence (II. 31) $GI : BD = \overline{GL}^2 : \overline{BF}^2$
But $GL$ is parallel to $BF$ (IV. 9); and the triangles $AGL$, $ABF$, being mutually equiangular, are similar (II. 20);
hence $GL : BF = AL : AF = (39)\ AM : AN$
or (Pn. 22) $\overline{GL}^2 : \overline{BF}^2 = \overline{AM}^2 : \overline{AN}^2$
Hence (Pn. 11) $GI : BD = \overline{AM}^2 : \overline{AN}^2$

**75.** With the same notation as in (72),
by (51) $V : v = \frac{1}{3} B \times A : \frac{1}{3} b \times a$
or (Pn. 21) $V : v = B \times A : b \times a$

**76.** If $B = b$, the proportion in (75) becomes
$$V : v = A : a$$
or, if $A = a$, $V : v = B : b$

**77.** If, in Theorem IX., the upper base $GI$ moves away from $BD$ until $GI$ becomes zero, $GH + HI + IK + KL + LG = 0$, and Theorem IX. becomes the same as Theorem VIII.

**78.** For their edges and altitude (39), or any other homologous lines are proportional; hence they are similar (4).

**79.** For their intersection must be a common diameter (60); and the diameter bisects either circle (III. 40).

**80.** If one hemisphere turn upon the diameter until the bases of the two hemispheres coincide, their curved surfaces must also coincide, or there would be points in the one or the other surface unequally distant from the centre of the sphere, which is contrary to (54).

**81.** (Fig., 58.) If $CE$ is perpendicular to the plane of the small circle $ADF$, then $CEA$, $CED$, $CEF$ are right angles (IV. I); hence (II. 28)

$\overline{EA}^2 = \overline{CA}^2 - \overline{CE}^2$, $\overline{ED}^2 = \overline{CD}^2 - \overline{CE}^2$, $\overline{EF}^2 = \overline{CF}^2 - \overline{CE}^2$

Hence, as $CA = CD = CF$, $EA = ED = EF$

Therefore $E$ is the centre of the circle $ADF$ (III. 1).

**82.** (Fig., 58.) As $CE$, or the distance of all the small circles from the centre of the sphere, and $CF$, the radius of the sphere, are constant, $EF$, or the radius of the small circles, must be constant (I. 50); and circles having equal radii are equal; therefore small circles equally distant from the centre of a sphere are equal.

**83.** (Fig., 58.) Suppose another sphere intersecting the sphere whose centre is $C$, and let $A$, $B$, $D$ be three points in the surface of both of the intersecting spheres. Pass a plane through $A$, $B$, $D$ (IV. 4); the section made by this plane must at the same time be a circle of both spheres (58); but as the circumference of this circle coincides in three points it must coincide throughout and be one and the same circle; that is, the intersection of the surfaces of two spheres is the circumference of a circle.

**84.** (Fig., 58.) Let $A$ and $F$ be any two points on the surface of the sphere whose centre is $C$; a plane can be made to pass through the three points $A$, $F$, and $C$ (IV. 4); and the section will be a great circle (60), and the arc an arc of a great circle.

**86.** For every point of the plane, except the point of tangency, is without the sphere (85); hence the radius to this point is the shortest distance from the centre of the sphere to the plane, that is, is perpendicular to the plane (IV. 8).

**87.** The solid described by $ABK$ is a cone (33); by $KC$ a frustum of a cone (33, 36); as arc $AC = DF$, if we turn the lower half of the semicircle over on the point $G$, so that $F$ is on $A$, $D$ will be on $C$, and also $M$ on $L$ (I. 76); hence $LC = MD$; and as $LC$, $MD$ are both perpendicular to $AF$, they are parallel to each other (I. 18); and $LD$ is a rectangle (I. 65); hence the solid described by $LD$ is a cylinder (12).

**88.** Place $R$ at the point where the perpendicular from $G$ meets $AB$, and $S$ where the perpendicular from $R$ meets $AF$; then we have a triangle $GSR$ similar to $KAB$ (II. 21);

hence $AB : AK = GR : SR =$ (III. 44) $GO : SR$

or (III. 28) $AB : AK =$ circ. $GO$ : circ. $SR$

hence $AB \times$ circ. $SR = AK \times$ circ. $GO$

But the surface described by $AB$ is the convex surface of a cone (33), and by (42) is equal to

$$AB \times \text{circ. } SR = AK \times \text{circ. } GO$$

**89.** If a perpendicular is drawn from $G$ to $CD$, it will be equal to $GO$ (III. 44); but as $LD$ is a rectangle (87), it will also be equal to $MD$. The surface described by $CD$ is the convex surface of a cylinder (87), the radius of whose base is $MD = GO$ and whose altitude is $LM$. Hence (15) the surface described by $CD = LM \times$ circ. $GO$.

**91.** (Fig., 61.) The surface described by a chord $BC$, as the semicircle revolves on $AF$, is equal to its altitude multiplied by circ. $GO$ (61); and this would be true whatever the length of the chord; it is true, therefore, if the chord is infinitely small; in this case the chord would coincide with its arc (III. 25), and the perpendicular from the centre $G$ to the chord would be the radius of a great circle; therefore the surface described by all the infinitely small chords that make up the arc $BC$, that is, by the arc $BC$ itself, is equal to the sum of their altitudes multiplied by the circumference of a great circle. But the sum of the altitudes of these small arcs is equal to $KL$, the altitude of the zone described by the arc $BC$; therefore the area of any zone is equal to the product of its altitude multiplied by the circumference of a great circle.

**92.** If $Z$ and $A$ represent, respectively, the area and the altitude of a zone, and $z$ and $a$ the area and altitude of any other zone on the same or an equal sphere, and $C$ the circumference of a great circle,

by (91) $Z = A \times C$, and $z = a \times C$

hence $Z : z = A \times C : a \times C = A : a$

**93.** By (62) $S = 4 \pi R^2$, and by (III. 32) $A = \pi R^2$;

hence $S = 4 A$

**95.** By (15) the convex surface of a cylinder $= 2 \pi R A$; but the radius of the base of the circumscribed cylinder is equal to the radius of the sphere, and its altitude to the diameter of the sphere; hence the convex surface of the circumscribed cylinder $= 2 \pi R \times 2 R = 4 \pi R^2$, which by (62) is also equal to the surface of the sphere.

**97.** A sector may be conceived to be composed of an infinite number of pyramids, whose vertices are at the centre of the sphere, and whose bases, being infinitely small planes, coincide with the surface of the zone. The altitude of each of these pyramids is the radius of the sphere, and the sum of the surfaces of their bases is the surface of the zone forming its base. The volume of each pyramid (51) is the product of its base by one third of its altitude, that is, of the radius of the sphere; and the volume of all the pyramids, that is, of the spherical sector, is, therefore, the product of the surface of the zone forming its base by one third of the radius of the sphere.

**99.** (Fig., 61.) 1st. Draw the radius $GB$. The spherical segment described by $KAB$, the half of a circular segment, as the semicircle $ACD$ revolves about $AF$, or the spherical segment of one base, is evidently equal to the spherical sector, described by the circular sector $GAB$, minus the cone described by the right triangle $GKB$. The volume of the spherical sector can be found by (97), and of the cone by (52); hence the volume of the spherical segment of one base can be found.

2d. Draw the radius $GC$. The spherical segment described by $LKBOC$, as the semicircle revolves about $AF$, or the spherical segment of two bases, is evidently equal to the spherical segment of one base described at the same time by $LABC$, the half of a circular segment, minus the spherical segment of one base described by $KAB$, the half of a circular segment. The volume of these spherical segments of one base can be found as shown above; hence the volume of the spherical segment of two bases can be found.

**100.** By (26) the volume of a cylinder $= \pi R^2 A$; but the radius of the base of the circumscribed cylinder is equal to the radius of the sphere, and its altitude to the diameter of the sphere; hence the volume of the circumscribed cylinder $= \pi R^2 \times 2R = 2\pi R^3$. The volume of the sphere $= \frac{4}{3}\pi R^3$.

$$\therefore \text{Sphere} : \text{Cylinder} = \tfrac{4}{3}\pi R^3 : 2\pi R^3 = \tfrac{2}{3} : 1$$

**101.** The altitude of the cone, hemisphere, and cylinder is equal to the radius of their bases, or to the radius of the hemisphere; that is, $A = R$.

By (52) $\quad \text{Cone} = \frac{1}{3}\pi R^2 A = \frac{1}{3}\pi R^2 \times R = \frac{1}{3}\pi R^3$

By (64) $\quad \text{Hemisphere} = \frac{1}{2} \times \frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$

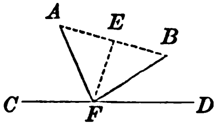
and by (26) $\quad \text{Cylinder} = \pi R^2 A = \pi R^2 \times R = \pi R^3$

$$\therefore \text{Cone} : \text{Hemisphere} : \text{Cylinder} = \tfrac{1}{3}\pi R^3 : \tfrac{2}{3}\pi R^3 : \pi R^3 = 1 : 2 : 3$$

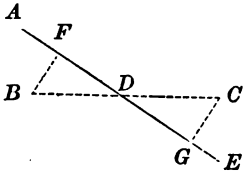
# BOOK VI.

## EXERCISES.

**55.** Join the given points $A$ and $B$; bisect $AB$ (1) with the perpendicular $EF$ meeting the given line $CD$ in $F$; join $AF$ and $BF$; $AF = BF$ (I. 53).

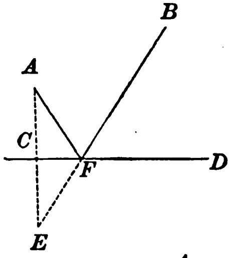


**56.** Let $A$ be the first point, $B$ and $C$ the two other given points. Join $BC$; bisect it (1) in $D$, and through $D$ draw $AE$. $AE$ is equally distant from $B$ and $C$. Draw $BF$, $CG$ perpendicular to $AE$ (2); the right triangles $BFD$, $DCG$ are mutually equiangular (I. 11; I. 35), and $BD = DC$; hence the triangles are equal, and $BF = CG$.

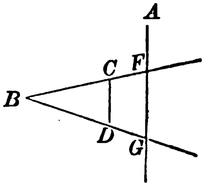


**57.** (Fig., 24.) At any point $E$, in the given line $AG$, draw $EC$ making angle $AEC =$ the given angle (5); through the given point $B$ draw $BD$ parallel to $CE$. $BD$ is the line required; for (I. 17) $ADB = AEC =$ the given angle.

**58.** From one of the given points $A$, draw $AC$ perpendicular to the given line $CD$; produce $AC$ to $E$ making $CE = AC$; join the other given point $B$ with $E$; join $A$ with $F$, the point of intersection of $BE$ and $CD$; angle $AFC =$ (I. 40) $CFE =$ (I. 11) $BFD$.



**59.** Cut off from the sides of the given angle $BC = BD$; join $CD$, and through the given point $A$ draw $AE$ parallel to $CD$ (6). Angle $BCD =$ (I. 42) $BDC =$ (I. 17) $BFE = BEF$.

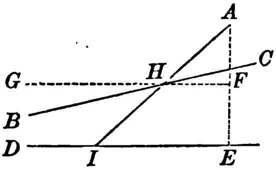


**60.** (Fig., 22.) Let $H$ be the first point, $D$ and $F$ the two other points. Join $DF$, and draw $HE$ bisecting $DF$ (1); through $H$ draw $GB$ perpendicular to $EH$ (3), and $DG$, $FB$ parallel to $EH$ (6). $GH = HB$; for, producing $FD$, $BG$ till they meet at $A$, in the triangle $ABF$, $GD$, $HE$ are parallel to the base $BF$; hence (II. 45), as $ED = EF$, $HG = HB$.

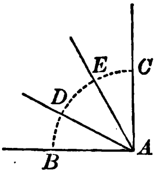
If $H$ is equally distant from $D$ and $F$, $BG$ and $FD$ will be parallel, and $HG = ED$, and $HB = EF$ (I. 62); hence, as $ED = EF$, $HG = HB$.

If the three points are in a straight line, the required parts become zero.

**61.** From the given point $A$ draw any line $AE$ meeting the given lines $BC$, $DE$; bisect $AE$ in $F$; through $F$ draw $FG$ parallel to $ED$ and cutting $BC$ in $H$; join $AH$ and produce it to $I$; as $AF = FE$, $AH = HI$ (II. 16).



**62.** At the vertex $A$ as a centre with any radius describe an arc cutting the sides in $B$ and $C$; with the same radius with $B$ and $C$ respectively as centres describe arcs cutting the arc $BC$ in $D$ and $E$; draw $DA$, $EA$, and these lines will trisect the angle $BAC$. For (III. 33) the arc $BE = DC = \frac{1}{6}$ of the circumference; hence the angle $BAE = DAC = \frac{1}{6}$ of four right angles, or $\frac{2}{3}$ of one right angle, and $BAD = EAC = DAE = \frac{1}{3}$ of a right angle.



**63.** (Fig., 49,) With $C$, one extremity of the given base $BC$, as a centre, with a radius $= BC$, describe an arc; from $B$ draw $BD$ tangent to this arc; make the angle $BCA = ABC$; produce $BD$ to meet $CA$; $BAC$ will be the triangle required. (For proof see Art. 49.)

**64.** (Fig., III. 22.) 1st. On the base $AC$ describe a segment $ADC$ of a circle that shall contain the given angle (20);

bisect the arc *ADC* at *D*, join *DA*, *DC*. *D* = the given angle; *DA* = *DC* (III. 12).

2d. (Fig., 10.) Let *AB* be the given base. At *A* and *B* make the angles *CAB*, *CBA* = the given angle (5), and *ACB* is the required triangle.

3d. Let *AC* be the given side. As the opposite angle = the adjacent angle next the base, at *A* make the angle *CAB* = the given angle; at *C* with a radius = *CA* describe an arc cutting *AB* in *B*; join *CB*.

4th. Let *CA* be the given side. At *C* make the angle *ACB* = the given angle; cut off *CB* = *CA*, and join *AB*.

If one angle of an isosceles triangle is given, the others can be found; for if the angle opposite the base is given, each of the angles at the base = $\frac{1}{2}(180° -$ the given angle$)$; if an angle at the base is given, the other is equal to it, and the angle opposite the base = 180° — 2 × the given angle. Hence all these cases can be constructed by (2).

**65.** (Fig., 31.) On the given base *AB* describe a segment *ADB* of a circle that shall contain the given angle (20); at any point *B*, of the base draw the perpendicular *BC* = the given altitude; through *C* draw *CD* parallel to *BA* intersecting the circumference in *D*; join *DA*, *DB*, and *ADB* will be the triangle required.

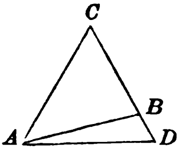
Impossible when *CD* does not cut the circumference; that is, when the altitude exceeds any perpendicular that can be drawn from the arc of the segment to the base.

**66.** (Fig., I. 72.) At *A*, on the given base *AC*, make the angle *A* = the given angle; make *AD* = the sum of the sides; join *DC*; at *C* make the angle *DCB* = *D*; then, as *BD* = *BC* (I. 45), *ABC* is the triangle required.

Impossible if the base is given > the sum of the sides (I. 69).

**67.** 1st. Through *B* on the given base *AB* draw an indefinite line *BC* making the angle *ABC* = the given angle (5); cut off from *CB* produced *BD* = the given difference; join *AD*;

draw $AC$ making the angle $DAC = CDA$; $ACB$ is the triangle required; for it has the given base $AB$, the given angle $ABC$ adjacent to the shorter side, and, as $CD = CA$ (I. 45), $AC - CB = CD - CB = BD$, the given difference.



2d. Construction same as above, except $BD$ is cut off between $B$ and $C$; and $BC - CA = BC - CD = BD$, the given difference.

Impossible if the base is given $<$ the difference of the sides (I. 69).

**68.** (Fig., I. 72.) From $C$ on the given base $AC$ draw an indefinite line $CB$, making the angle $ACB = \frac{1}{2}$ the given difference of the angles; with $A$ as a centre, and a radius = the given difference of the sides, describe an arc cutting $CB$ in $B$; join $AB$, and produce $AB$ indefinitely; from $C$ draw $CD$, making the angle $BCD = DBC$; a triangle $ADC$ will be constructed which is the triangle required. For $AC$ is the base, and as $DB = DC$ (I. 45), $DA - DC = DA - DB = BA =$ the given difference of the sides; moreover (I. 39)

$$\text{Angle } A + BCA = CBD = DCB$$

adding $BCA$ to each member,

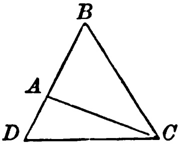
$$\text{Angle } A + 2\,BCA = DCB + BCA = DCA$$

but $2\,BCA =$ the given difference of the angles.

**69.** (Fig., I. 72.) Make the angle $D = \frac{1}{2}$ the given angle; cut off $DA =$ the given sum of the sides; with $A$ as a centre, with a radius = the given base, describe an arc cutting $DC$ in $C$ and join $AC$; draw $CB$ making the angle $DCB = D$, and $ABC$ is the triangle required. For $AC =$ the given base; as $BD = BC$ (I. 45), $AB + BC = AB + BD = AD =$ the given sum of the sides; and (I. 39) the angle $ABC = BCD + D = 2\,D =$ the given angle.

Impossible if the base is given $>$ the sum of the sides; or $<$ the perpendicular from $A$ to $DC$, that is, if the angle is given too obtuse.

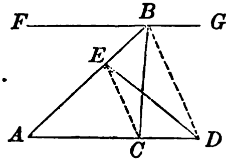
(Or (Fig., III. 22), on the given base $A\,C$ describe a segment $A\,D\,C$ that will contain the given angle (20); with $A$ and $C$ as foci, and the given sum of the sides as the transverse axis, describe an ellipse; where the two curves intersect will be the vertex of the required triangle.)



**70.** Make the angle $D = \frac{1}{2}$ the supplement of the given angle; cut off $D\,A =$ the given difference; with $A$ as a centre, with a radius $=$ the given base, describe an arc cutting $D\,C$ in $C$; draw $C\,B$, making the angle $D\,C\,B = C\,D\,B$, and $A\,B\,C$ is the triangle required. For $A\,C$ $=$ the given base; as $B\,C = B\,D$ (I. 45) $B\,C - B\,A = B\,D - B\,A = A\,D =$ the given difference; and (I. 34) the angle $B =$ the supplement of $(D + B\,C\,D) = 2\,D =$ the given angle.

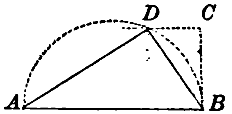
This problem is impossible if the base is given $<$ the difference of the sides.

(Or (Fig., III. 22), on the given base $A\,C$ describe a segment $A\,D\,C$ that will contain the given angle (20), with $A$ and $C$ as foci and the given difference as the transverse axis describe an hyperbola; where the two curves intersect will be the vertex of the required triangle.)

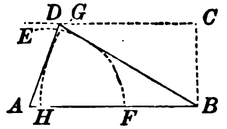


**71.** Let $A\,D$ be the given base, and $A\,B\,C$ the given triangle. Join $B\,D$; through $C$ draw $C\,E$ parallel to $D\,B$; join $E\,D$, and $A\,E\,D$ is the triangle required. For (II. 12) the triangle $E\,B\,C = E\,D\,C$; therefore $A\,B\,C = A\,E\,D$.

**72.** (Fig., 71.) Let $A\,E\,D$ be the given triangle. Draw $F\,G$ parallel to $A\,D$ at a distance equal to the given altitude, meeting $A\,E$, or $A\,E$ produced, in $B$; join $B\,D$; through $E$ draw $E\,C$ parallel to $B\,D$; join $B\,C$, and $A\,B\,C$ is the triangle required. For (II. 12) the triangle $E\,B\,C = E\,D\,C$; therefore $A\,B\,C = A\,E\,D$.

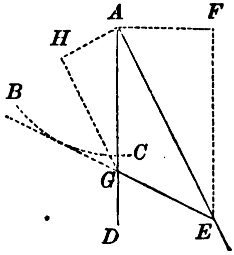


**73.** At any point *B* of the side *A B*, to which the given perpendicular is drawn, draw a perpendicular *B C* equal to the given perpendicular; through *C* draw *C D* parallel to *B A*; with *A* as a centre, and a radius equal to the other given side, describe an arc cutting *CD* in *D*; join *D A*, *D B*, and *A D B* is evidently the triangle required.



**74.** 1st. At any point *B* of the given side *A B* draw a perpendicular *B C* equal to the perpendicular that falls on the given side; through *C* draw *C D* parallel to *B A*. With *A* (or *B*, if the other perpendicular falls from *B*) as a centre, and a radius equal to the other perpendicular, describe an arc *E F*; from *B* draw *B D* tangent to the arc *E F* (18) meeting *C D* in *D*; join *A D*, and *A D B* is evidently the triangle required.

2d. With *A* and *B*, the extremities of the given side *A B*, as centres, and radii respectively equal to the perpendiculars from the vertices *A* and *B*, describe the arcs *E F*, *G H*; from *A* draw *A D* tangent to the arc *G H*, and from *B* draw *B D* tangent to the arc *E F* (18); *A D B* is evidently the triangle required.



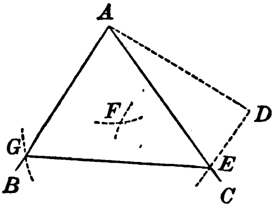
**75.** 1st. With the vertex *A* of the given angle as a centre, and a radius equal to the perpendicular from *A*, describe an arc *B C*; from any point in *A D* as *A* (or in *A E*, if the other perpendicular falls on *A E*) draw *A F* perpendicular to *A D* and equal to the other given perpendicular; through *F* draw *F E* parallel to *A D* meeting *A E* in *E*; from *E* draw a tangent to the arc *B C* (18) meeting *A D* in *G*; *G A E* is evidently the triangle required.

2d. At the vertex *A* of the given angle draw perpendicular to $AD$, $AF$ equal to the perpendicular falling on $AD$, and perpendicular to $AE$, $AH$ equal to the perpendicular falling on $AE$; through $F$ draw $FE$ parallel to $AD$ meeting $AE$ in $E$, and through $H$ draw $HG$ parallel to $AE$ meeting $AD$ in $G$; join $GE$, and $AGE$ is evidently the triangle required.

**76.** (Fig., 26.) Let $AB$, $BC$ be the segments of the base $AC$. On $AC$ describe a circular segment $ADC$ that shall contain the given angle (20); at $B$ draw $BD$ perpendicular to $AC$ meeting the arc in $D$; join $DA$, $DC$, and $ADC$ is evidently the triangle required.

**77.** (Fig., III. 22.) On $AC$, the given side, describe a circular segment $ADC$ that shall contain the given angle (20); with the middle of $AC$ as a centre, and a radius equal to the given bisecting line, describe an arc cutting the arc $ADC$ in $B$; join $BA$, $BC$, and $ABC$ is evidently the triangle required.

Impossible if the given bisecting line is $< \frac{1}{2}$ the base, or $>$ the perpendicular from the centre of $AC$ to the arc $ADC$.



**78.** At the vertex $A$ of the given angle draw perpendicular to $AB$ (or to $AC$, if the given perpendicular falls on $AC$) $AD$ equal to the given perpendicular; through $D$ draw $DE$ parallel to $AB$ meeting $AC$ in $E$; $E$ is the vertex from which the perpendicular falls; with $A$ and $E$ as centres with a radius equal to the radius of the circumscribed circle, describe arcs intersecting at $F$; and with the same radius and $F$ as a centre, describe an arc cutting $AB$ in $G$; join $GE$, and $AEG$ is evidently the triangle required.

Impossible if the radius of the circumscribed circle is not given $>$ half of any of the sides, and $<$ the given perpendicular.

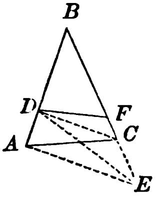
**79.** 1st. Join the given point $D$ with the opposite angle $C$; through $A$ draw $AE$ parallel to $DC$ meeting $BC$ produced in $E$; divide $BE$ at $F$ in the given ratio (22); join $DF$, and it will divide the triangle as required. For, drawing $DE$ (II. 13)

$$DBF : DFE = BF : FE$$

As the triangles $ADC$, $DCE$ have the same base $DC$, and the same altitude, viz. the distance between the parallels $DC$, $AE$, they are equivalent; hence

$$DFE = DFC + DCE = DFC + ADC = ADFC$$

and $\quad DBF : DFE = DBF : ADFC = BF : FE$



If the point $F$ does not fall between $B$ and $C$, draw the line parallel to $DC$ through $B$ instead of through $A$; or if $D$ is sufficiently near $A$ and the ratio sufficiently small, we may bring the point $F$ between $B$ and $C$ by inverting the given ratio.

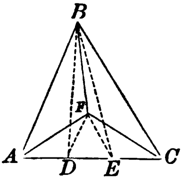
2d. (Fig., II. 20.) Let $m : n$ be the given ratio. Find a fourth proportional to $\sqrt{m+n}$, $\sqrt{n}$, and $AC$ (24); cut off from $AC$, $AH$ equal to this fourth proportional; through $H$ draw $HG$ parallel to $BC$, and the triangle will be divided as required. For by construction $\quad \sqrt{m+n} : \sqrt{n} = AC : AH$

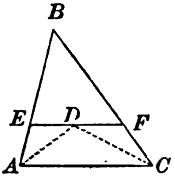
or (Pn. 22) $\quad m+n : n = \overline{AC}^2 : \overline{AH}^2 =$ (II. 29) $ABC : AGH$

Hence (Pn. 18)

$$m : n = ABC - AGH : AGH = GBCH : AGH$$

**80.** Trisect the base $AC$ of the given triangle $ABC$; from the points of division $D$, $E$ draw $DF$, $EF$ parallel respectively to $AB$, $CB$, and intersecting at $F$; join $FA$, $FB$, $FC$, and these lines will trisect the triangle $ABC$. Draw $BD$, $BE$. The triangles $ABD$, $DBE$, $EBC$ having equal bases $AD$, $DE$, $EC$, and the same altitude, are equivalent; hence each $= \frac{1}{3} ABC$; but the triangles $ABD$, $ABF$, having the same base $AB$ and the same altitude (the distance between the parallels $AB$, $DF$), are equivalent; hence $ABF = \frac{1}{3} ABC$; in like manner it can be proved that $FBC = \frac{1}{3} ABC$; hence also $AFC = \frac{1}{3} ABC$; and the triangle is trisected by lines drawn from a point within to the vertices.



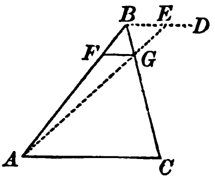


**81.** Bisect the angles $A$, $C$, of the given triangle $ABC$; through $D$, the point of intersection of the bisecting lines, draw $EF$ parallel to $AC$, and $EF$ is the line required; for (I. 17) the angle $EDA = DAC$
but $DAC = EAD$

$$\therefore EDA = EAD$$

and (I. 45) $ED = EA$
In like manner it can be proved that $DF = FC$;

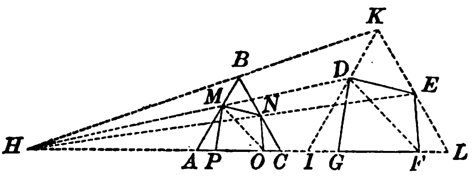
$$\therefore ED + DF = EA + FC$$



**82.** Through the vertex $B$ of the given triangle $ABC$ draw $BD$ parallel to $AC$ (6), and cut off $BE = BA - BC$; join $EA$ cutting $BC$ in $G$; through $G$ draw $GF$ parallel to $CA$; $FG$ is the line required; for, as $FG$ and $AC$ are parallel,
by (II. 17) $BA : BC = FA : GC$
by (Pn. 18) $BA - BC : BA = FA - GC : FA$
and (Pn. 15)

$$BA - BC : FA - GC = BA : FA = \text{(II. 20)}\ BE : FG$$

But, by construction, $BE = BA - BC$
hence $FG = FA - GC$



**83.** Let $DF$ be the given quadrilateral, $ABC$ the given triangle. Place $DF$, $ABC$ so that the bases shall be in the same straight line $AF$; through $D$ and $E$ draw $KI$, $KL$ parallel respectively to $BA$, $BC$; join $KB$ and produce $KB$ and $FA$ till they meet in $H$; join $HD$, $HE$, cutting $AB$, $BC$ in $M$ and $N$; join $MN$, and draw $MP$, $NO$ parallel respectively to $DG$, $EF$; $MO$ is the required quadrilateral. For, as its sides are parallel respectively to the sides of $DF$, $DF$ and $MO$ are mutually equiangular; and

by (II. 20) $MN : DE = HN : HE$

and $NO : EF = HN : HE$

hence (Pn. 11) $MN : DE = NO : EF$

and therefore, drawing the diagonals $MO$, $DF$, the triangles $MNO$, $DEF$, are similar (II. 23), and the angle $NMO = EDF$ and $NOM = EFD$; but the whole angle $NMP = EDG$, and $NOP = EFG$; hence the angle $PMO = GDF$, and $MOP = DFG$, and the triangles $MOP$, $DFG$ are also similar (II. 20); hence (II. 75) the quadrilaterals $DF$, $MO$ are similar.

Unless the triangle and quadrilateral can be so placed that the angle $A < G$ and $C < F$, the quadrilateral cannot be inscribed.

**84.** (Fig., 26.) On $AC$, a side of the given square, as a diameter describe a semi-circumference $ADC$. Fasten the ends of a string whose length is equal to the given line, one end at $A$, one at $C$; place a sharp point within the string, and drawing the string tight, move the point till it cuts the arc $ADC$, as at $D$; draw $AD$, $DC$, and these will be equal to the required parts of the given line. For $ADC$ is right-angled at $D$ (III. 23), and therefore (II. 27)

$$\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2$$

**85.** 1st. (Fig., I. 62.) As the three sides are given we can construct the triangle $ABD$ (8); with $B$ and $D$ as centres, and radii respectively equal to $AD$ and $AB$, describe arcs intersecting at $C$; join $BC$, $CD$, and $ABCD$ is the required parallelogram (I. 82).

2d. (Fig., I. 85.) With the given side $AB$ and half of each of the given diagonals (I. 85) construct the triangle $ABE$ (8); produce $AE$, $BE$ to $C$ and $D$ making $EC = AE$ and $ED = BE$; join $BC$, $CD$, $DA$, and $ABCD$ is the required parallelogram (I. 86).

3d. (Fig., I. 85.) Draw the two diagonals *A C*, *B D*, at the given angle bisecting each other; join *A B*, *B C*, *C D*, *D A*, and *A B C D* is the required parallelogram (I. 86).

4th. Subtract the given side from half the perimeter, and the remainder is the adjacent side. Draw these sides at the given angle, and complete the parallelogram as in case 1st.

**86.** (Fig., 45.) On the diagonal *A C* as a diameter describe a circle *A B C D*; draw the diameter *B D* at right angles to *A C*; join *A B*, *B C*, *C D*, *D A*, and *A B C D* is the required square (III. 23; III. 11; III. 12; I. 58).

**87.** (Fig., II. 10.) Draw *A D* equal to the base of the given triangle; draw an indefinite line *A B* making the given angle at *A*; at *A*, perpendicular to *A D*, draw *A E* equal to half the altitude of the given triangle; through *E* and *D* draw *E C* and *D C* parallel respectively to *A D* and *A B*; *A B C D* is evidently the parallelogram required (II. 10; II. 11).

**88.** (Fig., I. 54.) Draw the two sides *A B*, *A D* adjacent to the given angle so as to make the angle *A* equal to the given angle; with *B* as a centre, and a radius equal to the side adjacent to *A B*, and with *D* as a centre, and a radius equal to the remaining side, describe arcs intersecting at *C*; join *B C*, *C D*, and *A B C D* is evidently the required quadrilateral.

If $BC + CD < AB + AD$, the arcs described from *B* and *D* as centres may intersect in two points within the angle *A*; by joining either of these points with *B* and *D* we have a quadrilateral satisfying the conditions; that is, in this case we can have two different quadrilaterals satisfying the conditions.

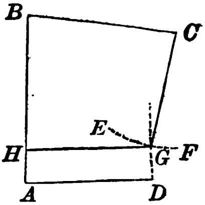
**89.** (Fig., I. 54.) 1st. Draw the sides *D A*, *A B*, *B C*, in the given order including at *A* and *B* the given angles; join *C D*, and *A B C D* is evidently the required quadrilateral.

2d. Draw the two known sides *A B*, *B C* that include one of the given angles, so as to make the angle *B* equal to this angle; at *C* draw an indefinite line *C D* making the angle *B C D* equal to the other given angle; with $A$ as a centre and a radius equal to the third given side, describe an arc cutting $CD$ in $D$; join $AD$, and $ABCD$ is evidently the required quadrilateral.

If $AD <$ than the diagonal from $A$ to $C$, but $>$ the perpendicular from $A$ to $DC$, the arc described from $A$ will cut $CD$ in two points; therefore, in this case, we may have two different quadrilaterals satisfying the conditions.

3d. One of the given angles $A$ will be included by known sides. Draw $AB$, $AD$ including the angle $A$; join $BD$; then we have two sides $BD$, $BC$, and the angle opposite $BD$, and constructing the triangle $BCD$ by (11), $ABCD$ will be the required quadrilateral.

There is the ambiguity in constructing the triangle $BCD$ that is explained in (11); so that in this case we may have two different quadrilaterals satisfying the conditions.



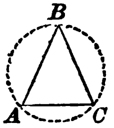
4th. At any points $A$, $B$, in the indefinite line $AB$, draw $AD$ and $BC$, make the angles at $A$ and $B$ equal to the given angles; make $AD$ equal to the side adjacent to the angle $A$, and $BC$ equal to the side adjacent to $B$; with $C$ as a centre, and a radius equal to the remaining side, describe an arc $EF$; through $D$ draw $DG$ parallel to $AB$ and cutting the arc $EF$ in $G$; through $G$ draw $GH$ parallel to $DA$, and join $GC$, and $BCGH$ is evidently the required quadrilateral.

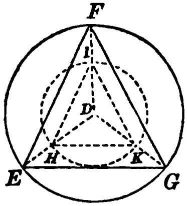
**90.** The fourth angle is equal to four right angles minus the sum of the three given angles (I. 67).

1st. (Fig., I. 54.) Draw the two given adjacent sides $AD$, $AB$, including the proper angle; at $B$ draw $BC$, making the angle $B$ equal to the other angle adjacent to the side $AB$, and at $D$ draw $DC$, making the angle $D$ equal to the remaining angle; $DC$, $BC$, intersecting at $C$, will complete the required quadrilateral.

2d. (Fig., 89.) Draw one of the given sides $BC$; at $B$ and $C$ draw the indefinite lines $BA$, $CG$, making the required angles with $BC$; at any point $A$, in $BA$, at the required angle with $BA$, draw $AD =$ the other given side; through $D$ draw $DG$ parallel to $AB$, intersecting $CG$ at $G$; through $G$ draw $GH$ parallel to $DA$; $BCGH$ is evidently the quadrilateral required.

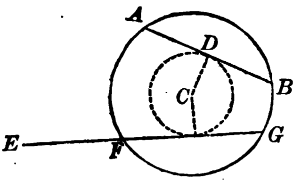
**91.** Circumscribe a circle (15) about the given triangle $ABC$; concentric with the given circle $EFG$, with a radius equal to the radius of the circle $ABC$, describe the circle $HIK$; make the arc $HI = AB$, $IK = BC$, and join $KH$, $HI$, $IK$; the triangle $HIK = ABC$ (III. 12; I. 48). Through the vertices $H$, $I$, $K$, draw the radii $DE$, $DF$, $DG$; draw $EF$, $FG$, $GE$, and $EFG$ is the required triangle. For as $HI$ divides the sides of the triangle $EFD$ proportionally, $HI$ is parallel to $EF$ (II. 18); in like manner $IK$, $KH$ are parallel respectively to $FG$, $GE$; hence $EFG$ is similar to $HIK$ (II. 21), and therefore to $ABC$.





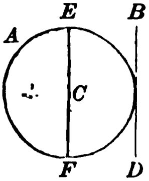
Or, if the angles $A$, $B$, $C$ are given, cut off the arcs $EF$, $FG$, containing respectively double the number of degrees in the angles $C$, $A$, and joining $EF$, $FG$, $GE$, $EFG$ would be the required triangle (III. 21; II. 20).

**92.** In the given circle whose centre is $C$ draw the chord $AB =$ the given line; draw $CD$ perpendicular to $AB$; with $C$ as a centre, with $CD$ as a radius, describe a circumference, to which draw $EG$ tangent (18); by (III. 44) $FG = AB =$ the given line.



**93.** (Fig., III. 16.) With the given points $A$ and $B$ respectively as centres, with the given radius, describe arcs; the circumference with the point of intersection $C$ (or $E$) of these arcs as a centre, with the radius *CA* (or *EA*), will be the circumference required.

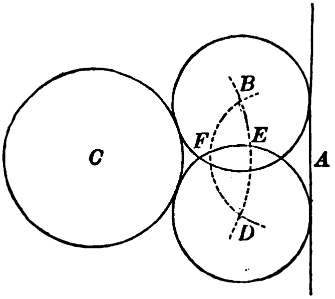
Impossible if the radius is given < ½ the distance *A* to *B*. If the radius is given = ½ the distance *A* to *B*, the centre of the required circle will be at the middle of the line *AB*, and in this case there can be but one circle; otherwise there may be two.



2d. Between the given line *BD* and the given point *A*, at a distance from *BD* equal to the given radius, draw *EF* parallel to *BD*; with *A* as a centre, with the given radius, describe an arc cutting *EF* in *C*; *C* is the centre of the required circumference.

If the radius is given = ½ the distance *A* to *BD*, the centre of the required circle will be at the middle of the perpendicular from *A* to *BD*, and in this case there can be but one circle, otherwise there may be two.

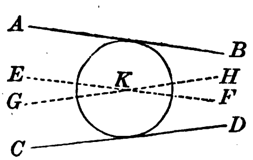
Impossible if the radius is given < ½ the distance *A* to *BD*.



3d. With the centre *C* of the given circle, with a radius equal to the sum of the radii of the given and the required circle, describe an arc *BED*; with the given point *A* as a centre, and a radius equal to the given radius, describe an arc *BFD* cutting *BED* in *B* and *D*; the circumference, with *B* or *D* as a centre, and the given radius, will be the circumference required (III. 51).

As the centre may be either at *B* or *D*, there can be two such circles; unless the given radius = ½ the distance of *A* from the circumference, in which case the centre will be in the line joining *AC*.

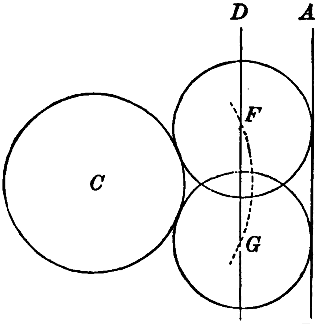
Impossible if the given radius is < ½ the distance from *A* to the given circumference. The given point must be without the given circumference.



4th. Between the given lines *A B*, *C D*, at a distance equal to the given radius, draw *E F*, *G H* parallel respectively to *A B*, *C D*, and intersecting in *K*; *K* will evidently be the centre of the required circumference.

Two such circles can be drawn, the other being at an equal distance beyond the point of intersection of *A B*, *C D*; unless the given lines are parallel and at the same time the given radius $= \frac{1}{2}$ the distance between the given lines, in which case there can be an infinite number of such circles.

Impossible if the given lines are parallel; unless the radius $= \frac{1}{2}$ the distance between the given parallel lines.

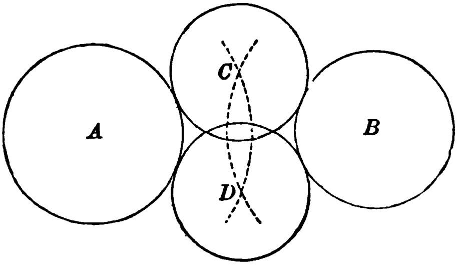


5th. Between the given line *A B* and the centre *C* of the given circle, at a distance equal to the given radius, draw *D E* parallel to *A B*; with the centre *C*, and a radius equal to the sum of the radii of the given and the required circle, describe an arc intersecting *D E* in *F* and *G*; the circumference, with *F* or *G* as its centre, and the given radius, will be the circumference required (III. 51).

As the centre may be either at *F* or *G*, there can be two such circles; unless the given radius $= \frac{1}{2}$ the distance of the line from the circumference of the circle, in which case the centre of the required circumference will be in the perpendicular from *C* to *A B*.

Impossible if the given radius $< \frac{1}{2}$ the distance of the line from the circumference of the given circle.

6th. With the centre of one of the given circles *A* as a centre, and a radius equal to the sum of the given radius and the radius of the circle *A*, describe an arc, and with the centre of the other given circle *B* as a centre, and a radius equal to



the sum of the given radius and the radius of the circle *B*, describe an arc cutting the first arc in *C*, *D*; the circumference whose centre is *C* or *D* is the circumference required (III. 51).

As the centre may be either *C* or *D*, two such circles can be drawn; unless the given radius = ½ the distance between the given circumferences, in which case the centre of the required circumference will be in the line joining *A* and *B*.

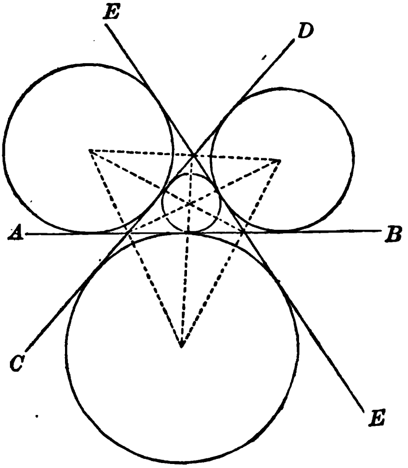
Impossible if the given radius < ½ the distance between the given circumferences, or if the two given circumferences are concentric, unless the given radius = ½ the distance between the circumferences.

**94.** (Fig., 14.) 1st. Join the given points *B*, *D*, with the line *BD*; bisect *BD* with a perpendicular *FC*, cutting the given line *EC* in *C*; *C* is the centre, and the line joining *CD* the radius, of the required circumference (I. 53).

2d. (Fig., 20.) At the given point *B* of the given line *DB*, draw *BC* perpendicular to *DB*; join the other given point *A* to *B* with the line *AB*; bisect *AB* with a perpendicular *EC*, cutting *BC* in *C*; *C* is the centre, and *CB* the radius, of the required circumference (I. 53; III. 47.)

3d. (Fig., 16.) Produce the given lines *CE*, *BF* till they meet at *A*; draw *AG* bisecting the angle *A* (4); from the given point *F*, in the line *FB*, draw *FD* perpendicular to *FB*, cutting *AG* in *D*; *D* is the centre, and *DF* the radius, of the required circumference. For, as shown in (16), a line bisecting an angle is equally distant from the sides of the angle, and hence *CE*, *BF* are tangents to the circumference whose radius is *DF* and centre *D*.

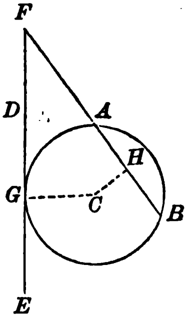
If $CE$, $BF$ are parallel, the centre of the required circumference will be at the middle of the perpendicular drawn from $F$ to $CE$.



4th. Produce the lines till they meet, and then proceed as in (16).

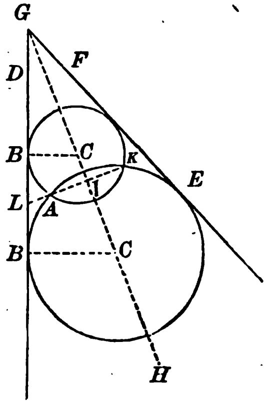
If no one of the lines is parallel to either of the others, four such circumferences can be drawn, of whose centres three will be at the intersections of the lines bisecting the exterior angles of the triangle formed by producing the given lines. If only two of the lines are parallel two such circumferences only can be drawn, one on either side of the line not parallel to either of the others. If the three lines are parallel, the problem is impossible.

5th. Through the given points $A$, $B$ draw a straight line $AB$ meeting the given line $DE$, or the given line produced, in $F$; find a mean proportional between $FA$, $FB$ (26), and cut off $FG$ equal to this mean proportional; $G$ will be the point of tangency of the required circumference (III. 64); at $G$ draw $GC$ perpendicular to $DE$, and bisect $AB$ with a perpendicular $HC$; the intersection of these two perpendiculars will be the centre (III. 47; III. 17), and $CG$ the radius of the required circumference.



If $AB$ is parallel to $DE$, the centre of the required circumference will be at the middle of the perpendicular drawn from the middle of $AB$ to $DE$.

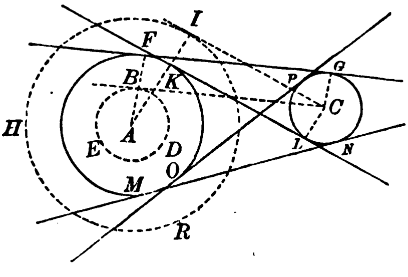
The problem is impossible if the given line passes between the given points.



6th. Produce the given lines *B D*, *E F* till they meet at *G*; draw *G H* bisecting the angle *G* (4); through the given point *A* draw *LK* perpendicular to *GH* (2), making *IK*=*AI*; find a mean proportional between *LA*, *LK* (26), and cut off *LB* equal to this mean proportional; *B* will be the point of tangency of the required circumference (III. 64); at *B* draw *B C* perpendicular to *B D*, and *C*, where it intersects *GH*, is the centre (III. 47; III. 17), and *CB* the radius of the required circumference.

As *L B* can be cut off on opposite sides of *L*, two such circumferences can be drawn. If *BD* is parallel to *EF*, the centre of the required circumference will be in the line drawn midway between, and parallel to, the given lines, and at a distance from the given point equal to half the distance between the given lines.

The problem is impossible unless the given point is between the given lines.



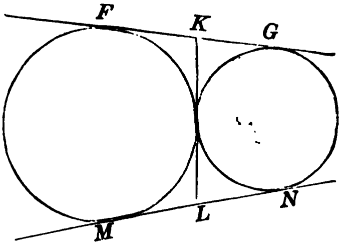
**95.** With the centre *A* of the greater circle, with a radius equal to the difference of the radii of the two given circles, describe a circumference *B D E*; from *C*, the centre of the other given circle, draw *C B* tangent to *B D E* (18); through *B* draw the radius *A F*, and through *C* the radius *C G* parallel to *A F*; join *F G*, and *F G* is tangent to the given circumferences. For, by construction, *B F* is equal and parallel to *C G*; hence *B G* is a parallelogram (I. 65); and as the angle at *B* is a right angle (III. 47), *B G* is a rectangle (I. 17; I. 62); hence *F G* is tangent to the given circumferences.

Or, when the circles are external to each other, with the centre *A*, with a radius equal to the sum of the radii of the given circles, describe a circumference *H I R*; from *C* draw *C I* tangent to *H I R*; draw the radius *A I*, cutting the circumference of the circle whose centre is *A* in *K*, and draw *C L* parallel to *I A*; join *K L*, and *K L* is a common tangent. For, in the same manner as above, *I L* can be proved to be a rectangle; hence *K L* is tangent to the given circumferences.

In like manner two other tangents *M N*, *O P* can be drawn; so that

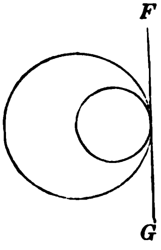
1st. When the circles are external to each other, we can have four tangents.

2d. If the circles move toward each other, when the circumferences touch externally, while *F G*, *M N* will still remain as tangents, *K L*, *O P* will coincide and form the same straight line whose point of tangency will be the point of tangency of the circumferences to each other; so that in this case we can have but three common tangents.



3d. If the circumferences cut, *K L* will disappear, but the two tangents *F G* and *M N*, or *F E* and *D B* in Fig., 94 Case 6, will still remain.

4th. If the circles move till the circumferences touch internally, *F G*, *M N* will coincide, and there can be but one common tangent, viz. that at the point of tangency of the two circumferences.



If the circles are equal, their circumferences will coincide and form one circumference to which can be drawn an infinite number of tangents.

5th. If the circles move so that one is within the other (in which case one circle must be less than the other), $FG$ will disappear. As there cannot be a tangent to a circumference from a point within (III. 9), there cannot be a common tangent.

# KEY

TO

# BRADBURY'S ELEMENTARY TRIGONOMETRY.

## ARTICLE 8.

| | | | | |
|---|---|---|---|---|
| 1. | | 3.575650 | 6. | 6425. |
| 2. | | 6.410946 | 7. | 66.07 |
| 3. | | 0.877717 | 8. | $959.83\dot{5}$ |
| 4. | | $\bar{3}.230449$ | 9. | 0.08129 |
| 5. | $\frac{4}{17} = 0.2352941$ | $\bar{1}.371610$ | 10. | 0.0007080344 |

## ARTICLE 10.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 347.676 | Log. | 2.541175 | 4. | 0.0004756 | Log. | $\bar{4}.677242$ |
| | 475.2 | " | 2.676876 | | 1355. | " | 3.131939 |
| | * 165215.6352 | " | 5.218051 | | .644438 | " | $\bar{1}.809181$ |
| 2. | 568. | Log. | 2.754348 | 6. | —0.017 | Log. | $\bar{2}.230449n$ |
| | 7496. | " | 3.874830 | | 25 | " | 1.397940 |
| | 846. | " | 2.927370 | | —165.4 | " | $2.218536n$ |
| | 1728. | " | 3.237544 | | 70.295 | " | 1.846925 |
| | 6224314285714. | " | 12.794092 | | | | |

| | | | |
|---|---|---|---|
| 7. | —14. | Log. | $1.146128n$ |
| | —7.643 | " | $0.883264n$ |
| | —0.004 | " | $\bar{3}.602060n$ |
| | —.428008 | " | $\bar{1}.631452n$ |

---

* This and some of the following answers cannot be *exactly* obtained from the Log. Tables. (See Art. 7.)

## ARTICLE 11.

| | | |
|---|---|---|
| 3. | .7478 | Log. $\bar{1}.873785$ |
| | 0.00456 | “ $\bar{3}.658965$ |
| | 163.99+ | “ 2.214820 |

| | | |
|---|---|---|
| 4. | 5000. | Log. 3.698970 |
| | 0.00149 | “ $\bar{3}.173186$ |
| | 3355704.7— | “ 6.525784 |

| | | |
|---|---|---|
| 5. | 0.00997 | Log. $\bar{3}.998695$ |
| | 64.16 | “ 1.807264 |
| | 0.00015539+ | “ $\bar{4}.191431$ |

| | | |
|---|---|---|
| 6. | —14.55 | Log. 1.162863$n$ |
| | 543. | “ 2.734800 |
| | —0.0267955+ | “ $\bar{2}.428063n$ |

| | | |
|---|---|---|
| 7. | —465. | Log. 2.667453$n$ |
| | —19.45 | “ 1.288920$n$ |
| | 23.9074+ | “ 1.378533 |

## ARTICLE 12.

| | | |
|---|---|---|
| 3. | 0.0176 | Log. $\bar{2}.245513$ |
| | | 4 |
| | 0.0000000959512576 | “ $\bar{8}.982052$ |

| | | |
|---|---|---|
| 4. | 1.179 | Log. 0.071514 |
| | | 9 |
| | 4.401765+ | “ 0.643626 |

| | | |
|---|---|---|
| 5. | —0.017 | Log. $\bar{2}.230449n$ |
| | | 3 |
| | —0.000004913 | “ $\bar{6}.691347n$ |

| | | |
|---|---|---|
| 6. | —14. | Log. 1.146128$n$ |
| | | 6 |
| | 7529536. | “ 6.876768 |

## ARTICLE 13.

| | | |
|---|---|---|
| 2. | 80.07 | Log. 3)1.903470 |
| | 4.31013— | “ .634490 |

| | | |
|---|---|---|
| 3. | 0.0764 | Log. 8)$\bar{8}+6.883093$ |
| | .72508+ | “ $\bar{1}.860386$ |

| | | |
|---|---|---|
| 4. | —17 | Log. 7)1.230449$n$ |
| | —1.49891+ | “ 0.175778$n$ |

| | | |
|---|---|---|
| 5. | —0.00496 | Log. 5)$\bar{5}+2.695482n$ |
| | —0.34601+ | “ $\bar{1}.539096n$ |

### ARTICLE 14.

| | | | |
|---|---|---|---|
| 2. | 416. | Log. comp. | 7.380907 |
| | 584. | " | 2.766413 |
| | 256. | " | 2.408240 |
| | 359.38+ | " | 2.555560 |

| | | | |
|---|---|---|---|
| 3. | 179. | Log. | 2.252853 |
| | 4968. | " | 3.696182 |
| | 489. | " comp. | 7.310691 |
| | 1818.552+ | " | 3.259726 |

| | | | |
|---|---|---|---|
| 4. | 1748. | Log. | 3.242541 |
| | 917. | " | 2.962369 |
| | 654. | " comp. | 7.184422 |
| | 513. | " " | 7.289883 |
| | 4.7776+ | " | 0.679215 |

| | | | |
|---|---|---|---|
| 5. $\sqrt{2.04}$ | Log. | = 0.309630 ÷ 2 = 0.154815 = Log. of | 1.428285 |
| $\sqrt{1.203}$ | " | = 0.080266 ÷ 2 = 0.040133 = " | 1.096813 |
| | | | 2.525098 |
| $\sqrt{0.1739}$ | Log. | = $\bar{1}.240300$ ÷ 2 = | $\bar{1}.620150$ |
| 331.9 | " comp. | | 7.478993 |
| $\overline{2.525098}^2$ | " " | = comp. of (0.402278 × 2) = | 9.195444 |
| 0.000197055 | " | | $\bar{4}.294587$ |

| | | | |
|---|---|---|---|
| 6. 23.3 | Log. | | 1.367356 |
| 6.764 | " | | 0.830204 |
| 85.31 | " | | 1.931000 |
| 253.4 | " comp. | | 7.596193 |
| $\sqrt[3]{47.64}$ | " " | = comp. of 1.677972 ÷ 3 = | 9.440676 |
| $\overline{2.768}^5$ | " " | = " 0.442166 × 5 = | 7.789170 |
| $\overline{9.853}^5$ | " | = 0.993568 × 5 = | 4.967840 |
| 9.97 | " comp. | | 9.001305 |
| 838.965+ | " | | 2.923744 |

7. 1. ; 2. ; 3. ; .5 ; 1.5 ; 0. ; $\bar{1}.5$ ; $\bar{1}.$ ; $\bar{2}.5$ ; —infinity.

9.

$$2048^x = 16$$

$$x \times \text{Log. } 2048 = \text{Log. } 16$$

$$x = \frac{\text{Log. } 16}{\text{Log. } 2048} = \frac{1.204120}{3.311330} = \frac{4}{11}$$

## ARTICLE 34.

| | | | |
|---|---|---|---|
| 3. | 9.357803 | 11. | 4° 32′ 29″ |
| 4. | 9.987242 | 12. | 16° 45′ 16″ |
| 5. | 8.571867$n$ | 13. | 53° 32′ 41″ |
| 6. | 9.998792$n$ | 14. | 7° 40′ 46″ |
| 7. | 10.507377 | 15. | 45° 24′ 52″ |
| 8. | 9.935218 | 16. | 15° 9′ 14″ |
| 9. | 9.982747 | 17. | 84° 17′ 22″ |
| 10. | 10.364786$n$ | | |

## ARTICLE 40.

(2.)

| | | | |
|---|---|---|---|
| R | 10. | R. | 10. |
| : $h$ 1676. | 3.224274 | : $h$ 1676. | 3.224274 |
| = sin. A 67° 13′ | 9.964719 | = cos. A 67° 13′ | 9.587989 |
| : $a$ 1545.23 | 3.188993 | : $b$ 649.03 | 2.812263 |

(3.)

| | | | |
|---|---|---|---|
| R | 10. | R | 10. |
| : $h$ 78.4 | 1.894316 | : $h$ 78.4 | 1.894316 |
| = cos. B 15° 51′ | 9.983166 | = sin. B 15° 51′ | 9.436353 |
| : $a$ 75.42 | 1.877482 | : $b$ 21.41 | 1.330669 |

## ARTICLE 41.

(2.)

| | | | |
|---|---|---|---|
| sin. B 75° 49′ comp. | 0.013445 | R | 10. |
| : $b$ 1075. | 3.031408 | : $h$ | 3.044853 |
| = R | 10. | = cos. B 75° 49″ | 9.389211 |
| : $h$ 1108.79 | 3.044853 | : $a$ 271.68 | 2.434064 |

(3.)

| | | | |
|---|---|---|---|
| cos. A 47° 31′ comp. | 0.170455 | R | 10. |
| : $b$ 17.45 | 1.241795 | : $h$ | 1.412250 |
| = R | 10. | = sin. A 47° 31′ | 9.867747 |
| : $h$ 25.84 | 1.412250 | : $a$ 19.05 | 1.279997 |

## ARTICLE 42.

### (2.)

| | | | |
|---|---|---|---|
| $h$ 172.8 | comp. 7.762456 | R | 10. |
| : R | 10. | : $h$ 172.8 | 2.237544 |
| = $b$ 14.17 | 1.151370 | = cos. B 4° 42′ 13″ | 9.998535 |
| : sin. B 4° 42′ 13″ | 8.913826 | : $a$ 172.218 | 2.236079 |

A = 90° — 4° 42′ 13″ = 85° 17′ 47″

## ARTICLE 43.

### (2.)

| | | | |
|---|---|---|---|
| $b$ 14. | comp. 8.853872 | sin. A 85° 45′ 49″ comp. | 0.001188 |
| : R | 10. | : $a$ 189. | 2.276462 |
| = $a$ 189. | 2.276462 | = R | 10. |
| : tan. A 85° 45′ 49″ | 11.130334 | : $h$ 189.518 | 2.277650 |

B = 90° — 85° 45′ 49″ = 4° 14′ 11″

## ARTICLE 44.

### (1.)

| | | | |
|---|---|---|---|
| $h$ 915. | comp. 7.038579 | R | 10. |
| : R | 10. | : $h$ 915. | 2.961421 |
| = $b$ 217. | 2.336460 | = cos. B 13° 43′ 8″ | 9.987430 |
| : sin. B 13° 43′ 8″ | 9.375039 | : $a$ 888.896 | 2.948851 |

A = 90° — 13° 43′ 8″ = 76° 16′ 52″

### (2.)

| | | | |
|---|---|---|---|
| $b$ 1927. | comp. 6.715118 | sin. A 5° 9′ 34″ comp. | 1.046107 |
| : R | 10. | : $a$ 174. | 2.240549 |
| = $a$ 174. | 2.240549 | = R | 10. |
| : tan. A 5° 9′ 34″ | 8.955667 | : $h$ 1934.89 | 3.286656 |

### (3.)

| | | | |
|---|---|---|---|
| sin. A 15° 39′ comp. | 0.569022 | R | 10. |
| : $a$ 17.94 | 1.253822 | : $h$ | 1.822844 |
| = R | 10. | = cos. A 15° 39′ | 9.983594 |
| : $h$ 66.503 | 1.822844 | : $b$ 64.038 | 1.806438 |

(4.)

| | | | |
|---|---|---|---|
| R | 10. | R | 10. |
| : *h* 47.9 | 1.680336 | : *h* 47.9 | 1.680336 |
| = sin. A 59° 17′ | 9.934349 | = cos. A 59° 17′ | 9.708245 |
| : *a* 41.1798 | 1.614685 | : *b* 24.467 | 1.388581 |

(5.)

| | | | |
|---|---|---|---|
| *h* 744. | comp. 7.128427 | R | 10. |
| : R | 10. | : *h* 744. | 2.871573 |
| = *a* 298. | 2.474216 | = cos. A 23° 36′ 42″ | 9.962028 |
| : sin. A 23° 36′ 42″ | 9.602643 | : *b* 681.712 | 2.833601 |

B = 90° — 23° 36′ 42″ = 66° 23′ 18″

(6.)

| | | | |
|---|---|---|---|
| *b* 13.44 | comp. 8.871601 | sin. A 35° 57′ 32″ comp. | 0.231210 |
| : R | 10. | : *a* 9.75 | 0.989005 |
| = *a* 9.75 | 0.989005 | = R | 10. |
| : tan. A 35° 57′ 32″ | 9.860606 | : *h* 16.60 | 1.220215 |

B = 90° — 35° 57′ 32″ = 54° 2′ 28″

(7.)

| | | | |
|---|---|---|---|
| cos. A 34° 7′ 10″ comp. | 0.082038 | R | 10. |
| : *b* 0.02518 | $\bar{2}$.401056 | : *h* | $\bar{2}$.483094 |
| = R | 10. | = sin. A 34° 7′ 10″ | 9.748901 |
| : *h* 0.03042 | $\bar{2}$.483094 | : *a* 0.01706 | $\bar{2}$.231995 |

## ARTICLE 51.

(2.)

C = 180° — (95° 4′ + 25° 14′) = 59° 42′

| | | | |
|---|---|---|---|
| sin. C 59° 42′ | comp. 0.063790 | sin. C 59° 42′ | comp. 0.063790 |
| : sin. A 95° 4′ | 9.998300 | : sin. B 25° 14′ | 9.629721 |
| = *c* 49.17 | 1.691700 | = *c* 49.17 | 1.691700 |
| : *a* 56.727 | 1.753790 | : *b* 24.278 | 1.385211 |

## ARTICLE 53.

(2.)

| | |
|---|---|
| *a* 95.5 | comp. 8.019997 |
| : *c* 173.2 | 2.238548 |
| = sin. A 27° 4′ | 9.658037 |
| : sin. C 55° 36′ 47″, or 124° 23′ 13″ | 9.916582 |

$$B = 180^\circ - (27^\circ\,4' + 55^\circ\,36'\,47'') = 97^\circ\,19'\,13'', \text{ or}$$
$$= 180^\circ - (27^\circ\,4' + 124^\circ\,23'\,13'') = 28^\circ\,32'\,47''$$

| | | | | |
|---|---|---|---|---|
| sin. A 27° 4′ comp. | 0.341963 | or | sin. A 27° 4′ comp. | 0.341963 |
| : sin. B 97° 19′ 13″ | 9.996446 | | : sin. B 28° 32′ 47″ | 9.679310 |
| = *a* 95.5 | 1.980003 | | = *a* 95.5 | 1.980003 |
| : *b* 208.17 | 2.318412 | | : *b* 100.29 | 2.001276 |

## ARTICLE 54.

### (2.)

$$\tfrac{1}{2}(A + B) = \tfrac{1}{2}(180^\circ - 14^\circ\,30') = 82^\circ\,45'$$

| | |
|---|---|
| $b + a = 195.$ | comp. 7.709965 |
| $: b - a = 13$ | 1.113943 |
| $= \tan. \tfrac{1}{2}(B + A) = \tan. 82^\circ\,45'$ | 10.895458 |
| $: \tan. \tfrac{1}{2}(B - A) = \tan. 27^\circ\,39'\,23''$ | 9.719366 |

Hence $A = 55^\circ\,5'\,37''$, and $B = 110^\circ\,24'\,23''$

| | |
|---|---|
| sin. A 55° 5′ 37″ | comp. 0.086140 |
| : sin. C 14° 30′ | 9.398600 |
| = *a* 91. | 1.959041 |
| : *c* 27.783 | 1.443781 |

## ARTICLE 55.

### (1.)

| Triangle A B D | | Triangle B C D | |
|---|---|---|---|
| *c* 75. | comp. 8.124939 | *a* 125. | comp. 7.903090 |
| : R | 10. | : R | 10. |
| = A D 30.46295 | 1.483772 | = D C 104.537 | 2.019270 |
| : cos. A 66° 2′ 7″ | 9.608711 | : cos. C 33° 14′ 56″ | 9.922360 |

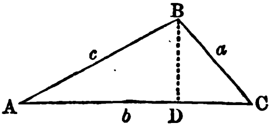
$$B = 180^\circ - (66^\circ\,2'\,7'' + 33^\circ\,14'\,56'') = 80^\circ\,42'\,57''$$

## ARTICLE 56.

### (2.)

| | | |
|---|---|---|
| *b* | 542. | comp. 7.266001 |
| : $c + a$ | 823. | 2.915400 |
| = $c - a$ | 129. | 2.110590 |
| : A D — D C | 195.88 | 2.291991 |

Hence A D = 368.94, and D C = 173.06



Triangle A B D

| | |
|---|---|
| $c$ 476. | comp. 7.322393 |
| : R | 10. |
| = A D 368.94 | 2.566956 |
| : cos. A $39^\circ 11' 14.5''$ | 9.889349 |

Triangle B C D

| | |
|---|---|
| $a$ 347. | comp. 7.459671 |
| : R | 10. |
| = D C 173.06 | 2.238196 |
| : cos. C $60^\circ 5' 2''$ | 9.697867 |

$$B = 180^\circ - (39^\circ\, 11'\, 14.5'' + 60^\circ\, 5'\, 2'') = 80^\circ\, 43'\, 43.5''$$

## MISCELLANEOUS EXAMPLES.

### (1.)

$$C = 180^\circ - (45^\circ\, 4' + 75^\circ\, 35') = 59^\circ\, 21'$$

| | |
|---|---|
| sin. C $59^\circ 21'$ comp. | 0.065351 |
| : sin. A $45^\circ 4'$ | 9.849990 |
| = $c$ 457. | 2.659916 |
| : $a$ 376.06 | 2.575257 |

| | |
|---|---|
| sin. A $45^\circ 4'$ comp. | 0.150010 |
| : sin. B $75^\circ 35'$ | 9.986104 |
| = $a$ | 2.575257 |
| : $b$ 514.48 | 2.711371 |

### (2.)

| | |
|---|---|
| $a$ 454 | comp. 7.342944 |
| : $c$ 753 | 2.876795 |
| = sin. A $45^\circ 25'$ | 9.852620 |
| : sin. C Impossible (Art. 53) | 10.072359 |

### (3.)

$$\tfrac{1}{2}(A + B) = \tfrac{1}{2}(180^\circ - 75^\circ\, 4') = 52^\circ\, 28'$$

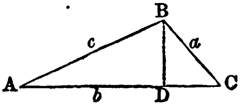
| | |
|---|---|
| $b + a = 146.$ | comp. 7.835647 |
| : $b - a =$ 32. | 1.505150 |
| = tan. $\tfrac{1}{2}(B + A) =$ tan. $52^\circ 28'$ | 10.114497 |
| : tan. $\tfrac{1}{2}(B - A) =$ tan. $15^\circ 55' 23''$ | 9.455294 |

Hence $A = 36^\circ\, 32'\, 37''$, and $B = 68^\circ\, 23'\, 23''$.

| | |
|---|---|
| sin. A $36^\circ 32' 37''$ | comp. 0.225166 |
| : sin. C $75^\circ 4'$ | 9.985079 |
| = $a$ 57. | 1.755875 |
| : $c$ 92.495 | 1.966120 |

### (4.)

| | |
|---|---|
| $b$ 74. | comp. 8.130768 |
| : $c + a$ 104. | 2.017033 |
| = $c - a$ 22. | 1.342423 |
| : A D — D C 30.919 | 1.490224 |



Hence A D = 52.4595, and D C = 21.5405.

| Triangle A B D | | Triangle B C D | |
|---|---|---|---|
| *c* 63. | comp. 8.200659 | *a* 41. | comp. 8.387216 |
| : R | 10. | : R | 10. |
| = A D 52.4595 | 1.719824 | = D C 21.5405 | 1.333256 |
| : cos. A 33° 37′ 26″ | 9.920483 | : cos. C 58° 18′ 22″ | 9.720472 |

$$B = 180° - (33° \, 37' \, 26'' + 58° \, 18' \, 22'') = 88° \, 4' \, 12''$$

(5.)

| | | | |
|---|---|---|---|
| *h* 75. | comp. 8.124939 | R | 10. |
| : R | 10. | : *h* 75 | 1.875061 |
| = *a* 35. | 1.544068 | = sin. B 62° 10′ 55″ | 9.946665 |
| : sin. A 27° 49′ 5″ | 9.669007 | : *b* 66.332 | 1.821726 |

$$B = 90° - 27° \, 49' \, 5'' = 62° \, 10' \, 55''$$

(6.)

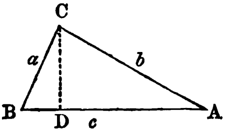
| | | | |
|---|---|---|---|
| R | 10. | R | 10. |
| : *h* 919. | 2.963316 | : *h* 919. | 2.963316 |
| = sin. A 37° 37′ | 9.785597 | = cos. A 37° 37′ | 9.898787 |
| : *a* 560.94 | 2.748913 | : *b* 727.95 | 2.862103 |

(7.)

| | | | |
|---|---|---|---|
| cos. A 34° 23′ | comp. 0.083400 | R | 10. |
| : *b* 45.3 | 1.656098 | : *h* | 1.739498 |
| = R | 10. | = sin. A 34° 23′ | 9.751839 |
| : *h* 54.89 | 1.739498 | : *a* 30.998 | 1.491337 |

(8.) Impossible (Art. 56).

| | |
|---|---|
| *c* 97. | comp. 8.013228 |
| : *b* + *a* 97. | 1.986772 |
| = *b* — *a* 17. | 1.230449 |
| : D A — B D 17. | 1.230449 |



Hence B D = 40, and D A = 57

| Triangle B C D | |
|---|---|
| *a* 40. | comp. 8.397940 |
| : R | 10. |
| = B D 40. | 1.602060 |
| : cos B 0° | 10.000000 |

Hence B C, C A fall on B A.

(9.)

| | |
|---|---|
| $a$ 0.05377 | comp. 11.269460 |
| : $b$ 0:06607 | $\bar{2}$.820004 |
| = sin. A 45° | 9.849485 |
| : sin. B 60° 19′ 34″, or 119° 40′ 26″ | 9.938949 |

C = 180° — (45° + 60° 19′ 34″) = 74° 40′ 26″, or
= 180° — (45° + 119° 40′ 26″) = 15° 19′ 34″

| | | | | |
|---|---|---|---|---|
| sin. A 45° | comp. 0.150515 | or | sin. A 45° | comp. 0.150515 |
| : sin. C 74° 40′ 26″ | 9.984274 | | : sin. C 15° 19′ 34″ | 9.422118 |
| = $a$ 0.05377 | $\bar{2}$.730540 | | = $a$ 0.05377 | $\bar{2}$.730540 |
| : $c$ 0.07334 | $\bar{2}$.865329 | | : $c$ 0.0201 | $\bar{2}$.303173 |

(10.)

| | |
|---|---|
| $b$ 35. | comp. 8.455932 |
| : $a$ 54. | 1.732394 |
| = sin. B 97° 15′ | 9.996514 |
| : sin. A Impossible (Art. 53) | 10.184840 |

Omitting Radius, or 10, in Art. 40, the calculation in Art. 94 is the same as in 40 ; 95 same as 41 ; 96 as 42 ; 97 as 43 ; 98 as 44.

The calculation in Art. 107 is the same as in 51 ; 109 same as 53 ; 110 as 54.

## ARTICLE 113.

| | | | |
|---|---|---|---|
| $s$ 682.5 | comp. 7.165897 | comp. 7.165897 | comp. 7.165897 |
| $s - a$ 335.5 | comp. 7.474307 | 2.525693 | 2.525693 |
| $s - b$ 140.5 | 2.147676 | comp. 7.852324 | 2.147676 |
| $s - c$ 206.5 | 2.314920 | 2.314920 | comp. 7.685080 |
| | 2)19.102800 | 2)19.858834 | 2)19.524346 |
| Log. tangents | 9.551400 | 9.929417 | 9.762173 |
| | ½ A 19° 35′ 37.24″ | ½ B 40° 21′ 51.76″ | ½ C 30° 2′ 31″ |
| | A 39° 11′ 14.5″ | B 80° 43′ 43.5″ | C 60° 5′ 2″ |

## ARTICLE 124.

(1.)

By (Art. 39, or 90) BC = AC tan. A

| | |
|---|---|
| AC 100. | 2. |
| tan. A 41° 15′ | 9.942988 |
| BC 87.6976 | 1.942988 |

## ARTICLE 125.

(1.)

| | |
|---|---|
| sin. D B C 54° | comp. 0.092042 |
| : sin. B D C 26° | 9.641842 |
| = D C 55. | 1.740363 |
| : B C 29.8 | 1.474247 |

## ARTICLE 126.

(2.)

| | | | |
|---|---|---|---|
| sin. A B D 3° comp. | 1.281200 | sin. B C D 99° comp. | 0.005380 |
| : sin. B A D 29° | 9.685571 | : sin. B D C 32° | 9.724210 |
| = A D 11.5 | 1.060698 | = B D | 2.027469 |
| : B D | 2.027469 | : B C 57.1556 | 1.757059 |

## ARTICLE 129.

(2.)

By (Art. 39, or 90) A C = A D tan. D    B C = A C tan. B A C

| | | | |
|---|---|---|---|
| A D 40 | 1.602060 | A C | 1.806271 |
| tan. D 58° | 10.204211 | tan. B A C 34° 45′ | 9.841187 |
| A C | 1.806271 | B C 44.4 | 1.647458 |

## ARTICLE 130.

(1.)

$$B = 180^\circ - (25^\circ + 92^\circ) = 63^\circ$$

| | |
|---|---|
| sin. B 63° | comp. 0.050119 |
| : sin. C 92° | 9.999735 |
| = A C 20. | 1.301030 |
| : A B 22.43 | 1.350884 |

## ARTICLE 132.

(1.)

$$\tfrac{1}{2}(A + B) = \tfrac{1}{2}(180^\circ - 39^\circ\ 25') = 70^\circ\ 17'\ 30''$$

| | |
|---|---|
| $BC + AC = 94.4$ | comp. 8.025028 |
| $: BC - AC = 5.6$ | 0.748188 |
| $= \tan. \tfrac{1}{2}(A + B) = \tan. 70^\circ\ 17'\ 30''$ | 10.445855 |
| $: \tan. \tfrac{1}{2}(A - B) = \tan. 9^\circ\ 24'\ 11''$ | 9.219071 |

Hence $A = 79^\circ\ 41'\ 41''$

| | |
|---|---|
| sin. A 79° 41′ 41″ | comp. 0.007062 |
| : sin. C 39° 25′ | 9.802743 |
| = B C. 50. | 1.698970 |
| : A B 32.268 | 1.508775 |

### ARTICLE 133.

(2.)

| | |
|---|---|
| Nat. cot. 48° 37′ | .88110 |
| Nat. cot. 54° 10′ | .72211 |
| | .15899 × 197 = 31.32103 ft. |

### ARTICLE 134.

(1.)

$ABD = 180° - (100° + 51°) = 29°$ $ACD = 180° - (120° + 55°) = 5°$

| | | | |
|---|---|---|---|
| sin. A B D 29° | comp. 0.314429 | sin. A C D 5° | comp. 1.059704 |
| : sin. B A D 100° | 9.993351 | : sin. C A D 55° | 9.913365 |
| = A D 40. | 1.602060 | = A D 40. | 1.602060 |
| : B D 81.25 | 1.909840 | : C D 375.95 | 2.575129 |

$BDC = ADC - ADB = 120° - 51° = 69°$

$\frac{1}{2}(DBC + BCD) = \frac{1}{2}(180° - 69°) = 55° 30'$

| | |
|---|---|
| C D + B D = 457.2 | comp. 7.339894 |
| : C D — B D = 294.7 | 2.469380 |
| = tan. ½(D B C + B C D) = tan. 55° 30′ | 10.162866 |
| : tan. ½(D B C — B C D) = tan. 43° 9′ 49″ | 9.972140 |

Hence B C D = 12° 20′ 11″

| | |
|---|---|
| sin. B C D 12° 20′ 11″ | comp. 0.670295 |
| : sin. C D B 69° | 9.970152 |
| = B D | 1.909840 |
| : B C 355.05 | 2.550287 |

### ARTICLE 136.

(1.)

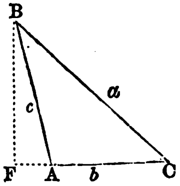
To find the angles of A B C.

*Geometrical Method.*

| | | |
|---|---|---|
| (Art. 55) | A C 38.62 | comp. 8.413188 |
| | : B C + B A 115. | 2.060698 |
| | = B C — B A 15. | 1.176091 |
| (Geom. II. 59) | : F C + F A 44.666 | 1.649977 |

$FA = \frac{1}{2}(44.666 - 38.62) = 3.023$

$FC = \frac{1}{2}(44.666 + 38.62) = 41.643$



| | | | |
|---|---|---|---|
| BA 50. | comp. 8.301030 | BC 65 | comp. 8.187087 |
| : R | 10. | : R | 10. |
| = FA 3.023 | 0.480438 | = FC 41.643 | 1.619542 |
| : cos. BAF $86^\circ 32' 2''$ | 8.781468 | : cos. C $50^\circ 9' 32''$ | 9.806629 |

$$BAC = 180^\circ - 86^\circ 32' 2'' = 93^\circ 27' 58''$$

$$ABC = 180^\circ - (93^\circ 27' 58'' + 50^\circ 9' 32'') = 36^\circ 22' 30''$$

### *Analytical Method.*

(Art. 111.) $\tan. \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; $\tan. \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

| | | |
|---|---|---|
| $s$ 76.81 | comp. 8.114582 | comp. 8.114582 |
| $s-a$ 11.81 | comp. 8.927750 | 1.072250 |
| $s-b$ 38.19 | 1.581950 | comp. 8.418050 |
| $s-c$ 26.81 | 1.428297 | 1.428297 |
| | 2)20.052579 | 2)19.033179 |
| Log. tangents | 10.0262895 | 9.5165895 |
| | $\frac{1}{2}$ A $46^\circ 43' 59''$ | $\frac{1}{2}$ B $18^\circ 11' 15''$ |
| | A $93^\circ 27' 58''$ | B $36^\circ 22' 30''$ |

To find AE. (Fig., Art. 135.)

Angle $ECA = BDA = 10^\circ$; $CAE = CDB = 12^\circ 14'$

Hence $AEC = 180^\circ - (10^\circ + 12^\circ 14') = 157^\circ 46'$

| | |
|---|---|
| sin. AEC $157^\circ 46'$ | comp. 0.422073 |
| : sin. ECA $10^\circ$ | 9.239670 |
| = AC 38.62 | 1.586812 |
| : AE 17.724 | 1.248555 |

To find the angle ABE. (Art. 54 or 110.)

$$BAE = BAC - EAC = 93^\circ 27' 58'' - 12^\circ 14' = 81^\circ 13' 58''$$

$$\tfrac{1}{2}(ABE + BEA) = \tfrac{1}{2}(180^\circ - EAB) = \tfrac{1}{2}(180^\circ - 81^\circ 13' 58'')$$
$$= 49^\circ 23' 1''$$

| | |
|---|---|
| $AB + AE = 67.724$ | comp. 8.169257 |
| $: AB - AE = 32.276$ | 1.508880 |
| $= \tan. \frac{1}{2}(BEA + ABE) = \tan. 49^\circ 23' 1''$ | 10.066715 |
| $: \tan. \frac{1}{2}(BEA - ABE) = \tan. 29^\circ 3' 41.7''$ | 9.744852 |

Hence $ABD = 20^\circ 19' 19.3''$.

To find AD, BD, CD.

$$DAB = 180^\circ - (BDA + ABD) = 180^\circ - (10^\circ + 20^\circ 19' 19.3'')$$
$$= 149^\circ 40' 40.7''$$

| | | | |
|---|---|---|---|
| sin. A D B 10° comp: | 0.760330 | sin. A D B 10° comp. | 0.760330 |
| : sin. A B D 20°19′19.3″ | 9.540700 | : sin. D A B 149°40′40.7″ | 9.703170 |
| = A B 50. | 1.698970 | = A B 50. | 1.698970 |
| : A D 100. | 2.000000 | : B D 145.37 | 2.162470 |

Angle D B C = A B C — A B D = 36° 22′ 30″ — 20° 19′ 19.3″
= 16° 3′ 10.7″

| | |
|---|---|
| sin. C D B 12° 14′ comp. | 0.673883 |
| : sin. D B C 16° 3′ 10.7″ | 9.441736 |
| = B C 65. | 1.812913 |
| : C D 84.827 | 1.928532 |

## Article 138.

(1.)

$(20 + 16) \times 2 \times 10 + 20 \times 16 \times 2 = 1360$ sq. ft. $= 151\frac{1}{9}$ sq. yds.

## Article 139.

(2.)

| | |
|---|---|
| 475. Log. | 2.676694 |
| 355. " | 2.550228 |
| 49° Log. sine | 9.877780 |
| 127263 Log. | 5.104702 |

127263 sq. feet = 2 acres, 3 roods, 27 sq. rods, $122\frac{1}{4}$ sq. feet.

## Article 141.

(1.)

| | |
|---|---|
| 75. Log. | 1.875061 |
| 7. " | 0.845098 |
| 71° Log. sine | 9.975670 |
| 4964 Log. | 2.695829 |

4964 square rods = 31 acres, 4 square rods.

## Article 145.

(1.)

Area trapezoid $= \frac{1}{2}(97 + 84) \times 47 = 4253.5$ square feet.

(2.)

Area trapezoid $= \frac{1}{2}(27 \times 77 \times \text{sin. } 93° + 28 \times 85 \times \text{sin. } 76° 15')$

| | | | |
|---|---|---|---|
| 27. Log. | 1.431364 | 28. Log. | 1.447158 |
| 77. " | 1.886491 | 85. " | 1.929419 |
| 93° Log. sine | 9.999404 | 76° 15′ Log. sine | 9.987372 |
| 2076.15 Log. | 3.317259 | 2311.79 Log. | 3.363949 |

Area trapezoid $= \frac{1}{2}(2076.15 + 2311.79)$ square rods
$=$ 13 acres, 2 roods, 33.97 square rods.

(3.)

(Art. 142.)

Area 1st triangle $= \sqrt{44.01 \times 8.21 \times 30.69 \times 5.11}$ square rods.

Area 2d triangle $= \sqrt{46.27 \times 10.43 \times 28.47 \times 7.37}$ square rods.

| | | | | | |
|---|---|---|---|---|---|
| 44.01 | Log. | 1.643551 | 46.27 | Log. | 1.665299 |
| 8.21 | " | 0.914343 | 10.43 | " | 1.018284 |
| 30.69 | " | 1.486997 | 28.47 | " | 1.454387 |
| 5.11 | " | 0.708421 | 7.37 | " | 0.867467 |
| | | 2)4.753312 | | | 2)5.005437 |
| 238.04 | " | 2.376656 | 318.21 | | 2.502718 |

Area trapezoid $= (238.04 + 318.21)$ square rods
$=$ 3 acres, 1 rood, 36.25 square rods.

## ARTICLE 146.

## MISCELLANEOUS EXAMPLES.

(1.)

| | | |
|---|---|---|
| (Fig., Art. 100.) | sin. C 15° comp. | 0.587004 |
| | : sin. A 7° | 9.085894 |
| | $= c$ 25 rods = 412.5 feet | 2.615424 |
| | : $a$ | 2.288322 |

(Art. 39 or 91.) $AD = c \times \cos. A$ $DC = a \cos. C$

| | | | |
|---|---|---|---|
| $c$ 25 rds. = 412.5 ft. | 2.615424 | $a$ | 2.288322 |
| cos. A 7° | 9.996751 | cos. C 15° | 9.984944 |
| A D 409.4 ft. | 2.612175 | D C 187.6 ft. | 2.273266 |

$AC = 409.4 + 187.6 = 597$ ft. No. pickets $= 597 \times 2 = 1194$.

(2.)

(Art. 39 or 90. Fig., Art. 120.) $BC = AC \tan. A$

| | |
|---|---|
| AC 160 | 2.204120 |
| tan. A 21° | 9.584177 |
| BC 61.418 | 1.788297 |

(3.)

| | |
|---|---|
| (Art. 128.) cot. 16° 25′ | 3.39406 |
| cot. 65° | 0.46631 |
| | 2.92775)25.00000(8.539 rods. |

(4.)

| | |
|---|---|
| (Art. 128.) cot. 19° 30′ | 2.82391 |
| cot. 21° 50′ | 2.49597 |
| | 0.32794)1.00000(3.049 miles. |

(5.)

| | |
|---|---|
| (Art. 133.) cot. 21° 30′ | 2.53865 |
| cot. 40° 10′ | 1.18474 |
| | 1.35391 × 65 = 88 feet. |

(Art. 39 or 90. Fig., Art. 120.) Angle $A = FBA = 40^\circ 10'$

$AC = BC \tan. B = BC \cot. A.$

| | |
|---|---|
| BC 65. | 1.812913 |
| cot. A 40° 10′ | 10.073622 |
| AC 77. | 1.886535 |

(6.)

(Art. 39 or 90.) $a = b \tan. A$ ; or $\tan. A = \frac{a}{b}$

| | |
|---|---|
| a 25. | 1.397940 |
| b 100. | comp. 8. |
| tan. A 14° 2′ 10″ | 9.397940 |

(7.)

Draw AE perpendicular to CD; join AC and AD.

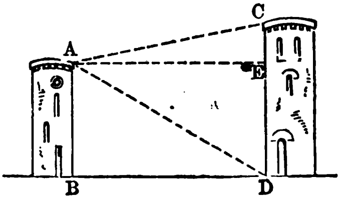
Angle $CAE = 10^\circ$

$ACD = 80^\circ$

$EAD = ADB = 30^\circ$

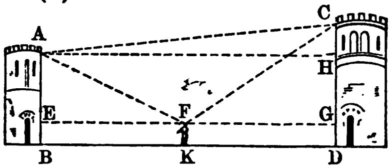
$CAD = CAE + EAD$

$= 10^\circ + 30^\circ = 40^\circ$



| | | | |
|---|---|---|---|
| sin. C A D 40° comp. | 0.191933 | A B = A D × sin. A D B. | |
| : sin. A C D 80° | 9.993351 | A D | 2.428322 |
| = C D 175. | 2.243038 | sin. A D B 30° | 9.698970 |
| : A D | 2.428322 | A B 134.06 | 2.127292 |

(8.)



Through A and F, respectively, draw A H and E G perpendicular to C D. join A C, A F, F C.

$$EF = BK = 76$$
$$CH = CD - HD = CD - AB = 64 - 50 = 14$$
$$AE = \sqrt{AF^2 - EF^2} = \sqrt{86^2 - 76^2} = 40.25$$
$$EB = FK = GD = AB - AE = 50 - 40.25 = 9.75$$
$$CG = CD - GD = 64 - 9.75 = 54.25$$
$$FG = \sqrt{FC^2 - CG^2} = \sqrt{97^2 - 54.25^2} = 80.41$$
$$AH = EG = EF + FG = 76 + 80.41 = 156.41$$
$$AC = \sqrt{AH^2 + CH^2} = \sqrt{156.41^2 + 14^2} = 157.035 \text{ ft.}$$

(9.)

As the angle B is so small, either A or C may be considered a right angle.

$$BC = AC \div \text{sin. } B$$

| | | |
|---|---|---|
| A C 3962. | | 3.597914 |
| sin. B 8.5776″ | comp. | 4.3810593 |
| B C 95273760.9 | | 7.9789733 |

(10.)

Angle A = $\frac{1}{2}(180° - 32') = 89° 44'$

| | | |
|---|---|---|
| sin. A 89° 44′ | comp. | 0.000005 |
| : sin. B 32′ | | 7.968870 |
| = B C | | 7.9789733 |
| : A C 886845.5 | | 5.9478483 |

(11.)

$$BC = AC \div \text{sin. } B$$

| | | |
|---|---|---|
| A C 3962. | | 3.597914 |
| sin. B 57′ 9″ | comp. | 1.779286 |
| B C 238341.8 | | 5.377200 |

### (12.)

Angle A = ½(180° — 31′ 7″) = 89° 44′ 26.5″

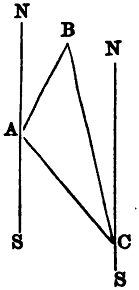
| | |
|---|---|
| : sin. A 89° 44′ 26.5″ | comp. 0.000005 |
| : sin. B 31′ 7″ | 7.956691 |
| = B C | 5.377200 |
| : A C 2157.23 | 3.333896 |

### (13.)

B C = A C ÷ sin. B

| | |
|---|---|
| A C, radius of earth's orbit, | 7.9789733 |
| sin. B 1″ | comp. 5.314425 |
| B C 19651597000000. | 13.2933983 |

### (14.)



Let A C be the impassable part of the line, and N S, N S the meridians through A and C.

Angle N A B = 20° N C A = 45° N C B = 24°

B C A = N C A — N C B = 21°

B = N A B + N C B = 44°

| | |
|---|---|
| sin. B C A 21° | comp. 0.445671 |
| : sin. B 44° | 9.841771 |
| = A B 20. | 1.301030 |
| : A C 38.77 | 1.588472 |

Whole length = 85 + 38.77 + 44 = 167.77

### (15.)

(Fig., Art. 134.) Angle A B D = 6° 30′ B A D = 107°

C A D = 45° A C D = 5° B D C = 63° 30′

| | | | |
|---|---|---|---|
| sin. A B D 6° 30′ | comp. 0.946141 | sin. A C D 5° | comp. 1.059704 |
| : sin. B A D 107° | 9.980596 | : sin. C A D 45° | 9.849485 |
| = A D 100. | 2. | = A D 100. | 2. |
| : B D 844.769 | 2.926737 | : C D 811.315 | 2.909189 |

½(B C D + D B C) = ½(180° — 63° 30′) = 58° 15′

| | |
|---|---|
| B D + C D = 1656.084 | comp. 6.780918 |
| : B D — C D = 33.454 | 1.524448 |
| = tan. ½(B C D + D B C) = tan. 58° 15′ | 10.208437 |
| : tan. ½(B C D — D B C) = tan. 1° 52′ 11″ | 8.513803 |

Hence B C D = 60° 7′ 11″

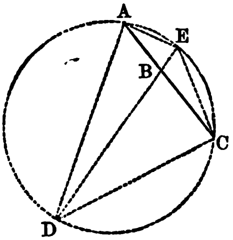
| | |
|---|---|
| sin. B C D 60° 7′ 11″ | comp. 0.061947 |
| : sin. B D C 63° 30′ | 9.951791 |
| = B D | 2.926737 |
| : B C 871.916 | 2.940475. |

### (16.)

Let D be my position, A the tower, B the steeple, and C the cliff; through the three points A, C, D describe a circumference A C D; produce B D to meet the circumference in E.

$$\text{Angle } ADE = ECA = 20° - 17° = 3°$$
$$EDC = EAC = 26° 30' - 20° = 6° 30'$$
$$AEC = 180° - (3° + 6° 30') = 170° 30'$$
$$AC = 25 + 54.35 = 79.35$$

| | |
|---|---|
| sin. A E C 170° 30′ | comp. 0.782391 |
| : sin. E C A 3° | 8.718800 |
| = A C 79.36 | 1.899602 |
| : A E 25.165 | 1.400793 |



| | |
|---|---|
| sin. A E C 170° 30′ | comp. 0.782391 |
| : sin. E A C 6° 30′ | 9.053859 |
| = A C 79.36 | 1.899602 |
| : E C | 1.735852 |

To find the angles A E B, B E C.

$$\tfrac{1}{2}(ABE + AEB) = \tfrac{1}{2}(180° - 6° 30') = 86° 45'$$

| | |
|---|---|
| A E + A B = 50.165 | comp. 8.299599 |
| : A E ∸ A B = 0.165 | $\bar{1}$.217484 |
| = tan. ½(A B E + A E B) = tan. 86° 45′ | 11.245773 |
| : tan. ½(A B E − A E B) = tan. 3° 18′ 54.3″ | 8.762856 |

Hence A E B = 83° 26′ 5.7″.

$$DEC = AEC - AEB = 170° 30' - 83° 26' 5.7''$$
$$= 87° 3' 54.3''$$

| | | | |
|---|---|---|---|
| sin. A D E 3° | comp. 1.281200 | sin. E D C 6° 30′ | comp. 0.946141 |
| : sin. A E D 83° 26′ 5.7″ | 9.997142 | : sin. D E C 87° 3′ 54.3″ | 9.999430 |
| = A E | 1.400793 | = E C | 1.735852 |
| : D A 477.68 | 2.679135 | : D C 480.2 | 2.681423 |

$$\text{Angle } DAC = DEC = 87° 3' 54.3''$$

| | |
|---|---|
| sin. A D B 3° | comp. 1.281200 |
| : sin. D A B 87° 3′ 54.3″ | 9.999430 |
| = A B 25. | 1.397940 |
| : D B 477.06 | 2.678570 |